SEMESTER 2 EXAMINATION 2009/10

CLASSICAL MECHANICS

Duration: 120 MINS

VERY IMPORTANT NOTE

Section A answers MUST BE in a separate blue answer book. If any blue answer booklets contain work for both Section A and B questions - the latter set of answers WILL NOT BE MARKED.

*Answer **all** questions in **Section A** and two **and only two** questions in **Section B**.*

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it. Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

A Sheet of Physical Constants will be provided with this examination paper.

An outline marking scheme is shown in brackets to the right of each question.

Only university approved calculators may be used.

Section A

A1. In the expression

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \sum_{i=1}^{N} \frac{1}{2}m_i\dot{\boldsymbol{\rho}}_i^2,$$

of the total kinetic energy T of a system of $i = 1, ...N$ particles, each of mass m_i, define all quantities appearing on the RHS and explain the significance of the two terms therein. Which one of the two is inertial frame dependent and which one is not? [6]

A2. Give the formula for the reduced mass of a system of two particles of masses m_1 and m_2. [2]

A3. What is the definition of a central force? Give any two examples of such forces. [3]

A4. State the three Kepler's laws of orbital motion. [3]

A5. The equation of motion of a particle of mass m moving near the surface of the Earth is

$$\ddot{\mathbf{r}} = -g\frac{\mathbf{r}}{r} - 2\omega \times \dot{\mathbf{r}} - \omega \times (\omega \times \mathbf{r})$$

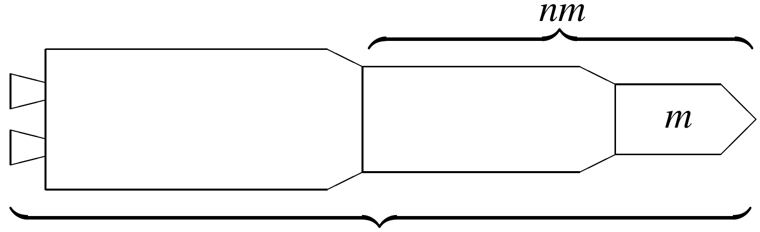
where $\mathbf{r}$ is the particle's position vector measured from the centre of the Earth in a frame rotating with the Earth at angular velocity ω. Identify and explain the significance of the terms on the RHS of this equation, with particular reference to the two terms containing cross products. [6]

Section B

B1. (a) A rocket in deep space burns fuel and reduces its mass from m_i to m_f. If the escaping gases have speed u relative to the rocket, show that the rocket's speed increases by

$$u \ln\left(\frac{m_i}{m_f}\right).$$ [7]

(b) A payload (for example a satellite) of mass m is mounted on a two stage rocket. The *total* mass of both rocket stages, fully fuelled, plus the payload, is Nm. The mass of the fully fuelled second stage plus payload, after first stage burnout and separation is nm. For both stages the ratio of burnout mass (casing) to initial mass (casing plus fuel) is r and the exhaust speed is u.



(i) Show that the speed gained from the first stage burn, starting from rest, is

$$v_1 = u \ln\left(\frac{N}{rN + n(1-r)}\right).$$ [4]

(ii) Show that the additional speed v_2, gained from the second burn, after separation of the burned out first stage is

$$v_2 = u \ln\left(\frac{n}{rn + 1 - r}\right).$$ [3]

(iii) Verify that the resulting payload velocity, $v_1 + v_2$, is maximised when $n = \sqrt{N}$. Show also that v_1 and v_2 are equal for this value of n. [6]

B2. (a) A uniform circular disc of mass $6m$ and radius a is free to turn in a horizontal plane about a fixed vertical axis through its centre. Calculate the moment of inertia of the disc about the vertical axis and define all quantities you introduce. [5]

(b) A mouse of mass m is standing on the disc at its rim, with the whole system at rest. The mouse marks its position on the edge of the disc. It sets off anticlockwise round the rim until it is diametrically opposite the initial mark and then runs radially inwards to radius $a/2$. Next it runs clockwise at constant radius until it is on the radius to the initial mark, at which point it turns and runs radially outwards to its starting point on the rim. Assume that the mouse always runs at the same speed.

Show that the absolute value of the angular displacement of the disc is

$$\frac{9\pi}{52} \text{ radians.}$$

(Hint: obtain the relation

$$\dot{\phi} = -\frac{1}{1 + 3a^2/r^2}\dot{\theta},$$

involving the anticlockwise displacement of the disc ϕ and that of the mouse relative to the disc θ, where r is the distance of the mouse from the center of the disc.) [13]

(c) Is this displacement clockwise or anticlockwise? [2]

B3. (a) Find from Newton's laws an expression relating the acceleration g due to gravity on the Earth's surface to Newton's gravitational constant G, the mass M_e and radius R_e of the Earth (assumed spherically symmetric). [3]

(b) Calculate the speed of a spacecraft in circular orbit about the Earth at a distance r_c from its centre. [4]

(c) The most energy efficient way of sending a spacecraft to the Moon is to boost its speed while it is in a circular orbit about the Earth such that its new orbit is an ellipse. The boost point is the perigee of the ellipse and the point of arrival at the Moon is the apogee. Calculate the percentage increase in speed required to achieve such an orbit. Assume that the spacecraft is initially in a low-lying circular orbit about the Earth (i.e., $r_c = R_e$). The distance between the Earth and the Moon is approximately $60R_e$.

(Hint: recall the generic orbit equation as a function of the eccentricity ϵ,

$$r = \frac{\alpha}{1 + \epsilon \cos\theta},$$

where r is the radius vector and

$$\alpha = \frac{ml^2}{k},$$

with $k = GM_e m$, m the spacecraft mass and ml its angular momentum. Assume that, for the polar angle, $\theta = 0$ corresponds to the perigee of the ellipse.) [13]

B4. Consider *small* transverse oscillations of a light string of length $6a$ stretched to tension T, carrying five beads, each of mass M. The beads are equally spaced, with each bead at distance a from either its nearest neighbours or the *fixed* ends of the string.

(a) If u_n denotes the (small) transverse displacement of the nth bead, show that the equation of motion for each bead is:

$$\ddot{u}_n = \frac{T}{Ma}(u_{n+1} - 2u_n + u_{n-1}).$$

[4]

(b) What conditions need to be imposed on u_0 and u_6 to make the equation of motion correct for the 1st and 5th beads? [2]

(c) Show that the normal mode angular frequencies, ω_m, for this system are given by

$$\omega_m^2 = \frac{4T}{Ma}\sin^2\left(\frac{m\pi}{12}\right),$$

where $m = 1, 2, 3, 4, 5$. [8]

(d) Why is it not necessary to consider other values of m? [2]

(e) Sketch the displacements of the beads in the five normal modes. [4]

END OF PAPER