SEMESTER 1 EXAMINATION 2019-2020

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct 'Word to Word' translation dictionary AND it contains no notes, additions or annotations.

Section A

A1. What is meant by a *central* force? Give two examples of such forces. [2]

Why can a central force between two objects not affect the angular momentum of one about the other? [2]

A2. State Kepler's laws and outline the physical assumptions that underly them. [4]

A3. Show that, if a spacecraft of total mass $m(t)$ propels itself by ejecting exhaust gas from its rocket motor with a relative velocity $\mathbf{u}$, then its velocity $\mathbf{v}(t)$ satisfies

$$m\mathrm{d}\mathbf{v} = -\mathbf{u}\mathrm{d}m,$$ [2]

and hence, making clear any assumptions in your derivation, that the initial and final speeds v_i and v_f are related to the initial and final masses m_i and m_f by

$$v_f = v_i + u \ln \frac{m_i}{m_f}.$$ [2]

A4. Instead of using jet thrusters to rotate a spacecraft, an engineer proposes using the reaction obtained when using an electric motor, attached to the spacecraft, to rotate a flywheel. Explain, with reference to physical laws, why this will work. What must be done to a flywheel with moment of inertia I_f in order to rotate the spacecraft of moment of inertia I_s through 90 degrees? [4]

A5. At a rifle range in Tasmania (41° S), a rifle bullet is fired with an initial speed (*muzzle velocity*) v horizontally towards the west. Explain how, and in which direction, it is deflected as a result of the Earth's rotation. [4]

Section B

B1. (a) Explain how a rotation angle and axis can be represented by a vector $\boldsymbol{\varphi}$. [2]

(b) Demonstrate, with an example, that rotations through finite angles do not in general commute – i.e., $\boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 \neq \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_1$. [1]

(c) Show that, if a fixed-length vector $\mathbf{A}$ rotates with angular velocity ω about an axis defined by the vector $\hat{\boldsymbol{\omega}}$, and we define $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{\omega}}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}.$$ [4]

A thin coin of radius a and mass m is set spinning upon a smooth table. After spinning on its edge for a while, the coin begins to topple, and establishes a motion in which its rotational symmetry axis $\hat{\boldsymbol{n}}$ precesses about the vertical $\hat{\boldsymbol{\Omega}}$ at a constant angle ϑ. The rolling motion is a combination of rotation at rate Ω about $\hat{\boldsymbol{\Omega}}$ and rotation at rate ω_n about $\hat{\boldsymbol{n}}$. As the coin does not slip where it touches the table, this corresponds instantaneously to rotation about a diameter $\hat{\boldsymbol{d}}$ through this point, where $\Omega\hat{\boldsymbol{\Omega}} - \omega_n\hat{\boldsymbol{n}} = \omega_d\hat{\mathbf{d}}$ and $\omega_n = \Omega\cos\vartheta$.

(d) Show that the coin's moment of inertia about $\hat{\mathbf{d}}$ is $ma^2/4$. [4]

(e) Show that the coin's angular momentum $\mathbf{L}$ satisfies

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \Omega\,\hat{\boldsymbol{\Omega}} \times \mathbf{L} = mga\cos\vartheta\,\hat{\mathbf{h}}$$

where Ω is the angular precession frequency, $\hat{\boldsymbol{\Omega}}$ is a vertical unit vector, and $\hat{\mathbf{h}}$ is a unit vector parallel to the horizontal diameter of the coin. [4]

(f) Given that the angular momentum may be written as $\mathbf{L} = I_d\omega_d\hat{\mathbf{d}}$, where ω_d the coin's instantaneous angular velocity, show that $\Omega\omega_d = 4g/a$. [3]

(g) By considering the triangle of angular velocity vectors, or otherwise, show that the coin therefore precesses with an angular velocity

$$\Omega = \sqrt{\frac{4g}{a\sin\vartheta}}.$$ [2]

TURN OVER

B2. (a) (i) Give an expression, in vector form, for the gravitational force $\mathbf{F}_{12}$ upon a body of mass m_1 at position $\mathbf{r}_1$, due to a second body of mass m_2 at position $\mathbf{r}_2$, in terms of the relative position $\mathbf{r}_{12} \equiv \mathbf{r}_1 - \mathbf{r}_2$. [2]

(ii) Hence show that the gradient of the magnitude of the gravitational force has a radial component

$$\frac{\mathrm{d}F_{12}}{\mathrm{d}r_{12}} = -\frac{2Gm_1m_2}{r_{12}^3}.$$ [2]

(b) A body in orbit about the Sun comprises two small discs, each of mass m, that are separated along their common axis by a distance d. The axis makes an angle ϑ to the plane of the orbit, and you may assume that $d \ll |\mathbf{r}_{12}|$, where $\mathbf{r}_{12}$ is the position of the body with respect to the Sun.

(i) Sketch the situation described, and show that the torque $\boldsymbol{\tau}$ acting upon the body will be

$$\boldsymbol{\tau} = \frac{GM_\odot md^2 \sin 2\vartheta}{2r_{12}^3},$$ [4]

where $r_{12} \equiv |\mathbf{r}_{12}|$ and $M_\odot$ is the mass of the Sun.

(ii) Hence show that, if the body has a moment of inertia I about the discs' common axis, and rotates about that axis with an angular velocity ω, its angular momentum vector will rotate with an angular velocity

$$\Omega = \frac{\tau}{I\omega\cos\vartheta},$$

and therefore that

$$\Omega = \frac{1}{\omega}\frac{GM_\odot}{r_{12}^3}\frac{md^2}{I}\sin\vartheta.$$ [6]

You may assume that Ω is small in comparison with both ω and the orbital angular velocity of the body about the Sun.

(iii) Indicate, with the aid of a sketch, the direction and path of this precession with respect to the body, its rotation axis, and the Sun. [1]

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(c) The Earth has a greater radius in its equatorial plane than along its polar axis. It may be modelled as a sphere of radius r_e, from which two discs of radius $\sqrt{2/5}\,r_e$, thickness $(r_e - r_p)$ and separation $2r_e$ have been subtracted, where $r_e = 6378$ km and $r_p = 6357$ km are the radii in the equatorial and polar directions. The density of the Earth near its surface is around 2750 kg m^{-3}, and the mean density of the Earth is 5514 kg m^{-3}. The mass of the Sun, $M_\odot = 1.989 \times 10^{30}$ kg, the distance of the Earth from the Sun is around 1.5×10^8 km, the gravitational constant $G = 6.674 \times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$, and the Earth's axis makes an angle $\vartheta = 66.6^\circ$ with its orbital plane.

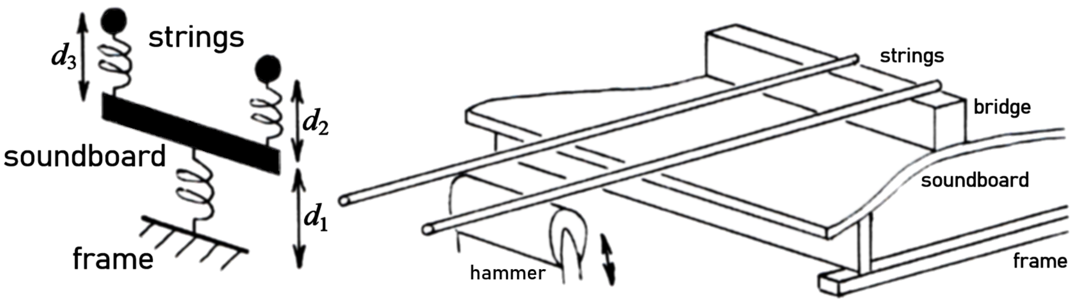
(i) Using the data given, calculate the mass of the Earth, $M_\oplus$. [2]

(ii) Assuming that the moment of inertia of the Earth is approximately given by $(2/5)M_\oplus r_e^2$, find the period with which the Earth's axis precesses, at the time of the summer solstice, due to the effect of the Sun's gravity gradient upon the non-symmetrical Earth. [3]

TURN OVER

B3. (a) Explain what is meant by (i) *simple harmonic motion* and (ii) the *normal mode* of an oscillating system. [4]

Each mid-range note of a piano is produced by a pair of identical strings that pass over the same bridge to transmit their motions to the soundboard. The system can be modelled as a pair of equal masses m, representing the strings, that are connected through springs of natural length l and spring constant k, representing the restoring mechanisms of the displaced strings, to a mass m_0 corresponding to the soundboard, which is itself attached by a spring of natural length l_0 and constant k_0 to a solid anchor representing the piano frame. The effects of gravity and sideways or tilting motions can be neglected.



To analyse the piano dynamics, define the displacement of the soundboard from rest as $x_1 \equiv d_1 - l_0$, and $x_2 \equiv d_2 - l$ and $x_3 \equiv d_3 - l$ as the displacements of the string-soundboard distances from their rest values.

(b) Setting out your working formally, derive the three equations of motion

$$\begin{aligned} mm_0\ddot{x}_1 &= -k_0 m x_1 + km(x_2 + x_3) \\ mm_0\ddot{x}_2 &= k_0 m x_1 - k(m + m_0)x_2 - kmx_3 \\ mm_0\ddot{x}_3 &= k_0 m x_1 - kmx_2 - k(m + m_0)x_3 \end{aligned}$$

where $\ddot{x}_1 \equiv \mathrm{d}^2 x_1/\mathrm{d}t^2$, etc. [4]

(c) By substituting the normal mode solutions $x_j = a_j \exp(\mathrm{i}\omega t)$, where $j = 1\ldots3$, and assuming that $k_0 \ll k$, show that the common frequency of motion ω must satisfy

$$mm_0\omega^2 \left(mm_0\omega^2 - km_0\right) \left[mm_0\omega^2 - k(2m + m_0)\right] = 0.$$ [7]

(d) Hence find expressions for the frequencies ω_{sym} and ω_{asym} of the *symmetric* and *asymmetric* modes, identifying which is which. [3]

(e) Describe and interpret the symmetric mode and, by considering the role of the soundboard in converting the string motion to sound, explain how the symmetric and asymmetric modes will compare in loudness and duration after the hammer has struck the strings. [2]

TURN OVER

B4. A comet of mass m moves in the gravitational field of a star of mass M, and its position is described by its polar coordinates (r, ϑ) relative to the star. The gravitational potential is given by $\mathcal{V}(r) = GMm/r$. Assume that $M \gg m$.

(a) Show that the angular momentum of the comet about the star will be $L = mr^2\dot{\vartheta}$, where $\dot{\vartheta}$ signifies $\mathrm{d}\vartheta/\mathrm{d}t$, the rate of change of ϑ with time. [2]

(b) Show that the comet's total energy $\mathcal{E}$ may be written as

$$\mathcal{E} = \frac{m}{2}\dot{r}^2 + \left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right) \equiv \frac{m}{2}\dot{r}^2 + \mathcal{U}(r),$$

where $\dot{r} \equiv \mathrm{d}r/\mathrm{d}t$ and $\mathcal{U}(r)$ is the effective potential in which the comet's radial motion occurs. [4]

(c) Assuming that the comet follows an elliptical orbit with the star at one focus, show from these results that the length $2a$ of the ellipse's major axis will be

$$2a = \frac{GMm}{-\mathcal{E}}.$$ [4]

(d) By differentiating the total energy with respect to the time t, derive the equation of radial motion of the comet,

$$\frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{L^2}{m^2r^3} - \frac{GM}{r^2}.$$ [2]

(e) By writing $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \dot{\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta} \equiv \frac{L}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\vartheta}$ and making the substitution $r \equiv 1/u$, show that the equation of motion may be rewritten as

$$\frac{\mathrm{d}^2u}{\mathrm{d}\vartheta^2} = -u + \frac{GMm^2}{L^2}.$$ [4]

(f) Hence show that the comet will trace out a path $r(\vartheta)$ of the form

$$r = \frac{L^2}{GMm^2\left(1 + \alpha\cos\vartheta\right)}$$

where

$$\alpha^2 = 1 + \frac{2L^2\mathcal{E}}{\left(GMm\right)^2 m}.$$ [4]

END OF PAPER

