(A1) Bookworm

For each particle: $\underline{F}_i = \dot{\underline{p}}_i$ where $\underline{p}_i = m_i \dot{\underline{r}}_i$

Summing: $\sum_i \underline{F}_i = \sum_i \dot{\underline{p}}_i = \dot{\underline{P}}$ where $\underline{P} = \sum_i \underline{p}_i$ define $\underline{P}$ [2]

$$\sum \underline{F}_i = \sum_i \underline{F}_i^{ext} + \sum_i \sum_{j \neq i} \underline{F}_{ij}$$

falls apart into sum over all pairs $\underline{F}_{ij} + \underline{F}_{ji} = 0$. [1]

$\therefore \sum \underline{F}_i = \sum \underline{F}_i^{ext} = \underline{F}^{ext}$

and $\boxed{\underline{F}^{ext} = \frac{d\underline{P}}{dt}}$ (*) [1]

[4]

(A2) Bookworm

IF the rotational motion takes place about an axis that traces the surface of a cone when it bodily turns, then the axis is said to precess around the vertical direction and the phenomenon is called precession. [1]

(Example: motion of a spinning top.)

[2]

[3]

(A3) Bookworm

Just relate two expressions for grav. force: [2]

$$mg = \frac{GMm}{r_e^2} \rightarrow \boxed{g = \frac{GM}{r_e^2}} \quad [1]$$

[3]

(A4)

Force is central [1] $\underline{F} = f(r)\,\underline{r}$ $\therefore \frac{d\underline{L}}{dt} = \underline{r} \times \dot{\underline{p}} = 0 \Rightarrow \underline{L} = \text{const.}$ [2]

But $\underline{L}$ always perp. to plane defined by $\underline{r}$ and $\underline{v}$, hence whole motion lies in a plane. [2]

[5]

(A5)

[2] $-\overline{\omega} \times (\overline{\omega} \times \overline{R})$

$\overline{\omega}$ = Earth's angular velocity
$\overline{R}$ = Earth's radius (outwards from centre) [1]

$$\frac{\omega^2 R}{g} \simeq 0.35\% \quad [2]$$

[5]

B1 (a) let ρ be mass per unit cross-sectional area:

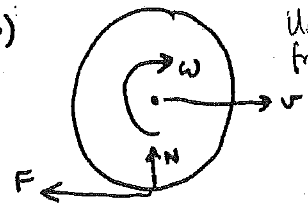
Then, [2]

$$m = \int_0^a \rho\, 2\pi r\, dr = \pi \rho a^2$$ → sum over thin cylindrical shells.

$$I = \int_0^a \rho\, 2\pi r . r^2\, dr = \frac{\pi \rho a^4}{2} = \boxed{\tfrac{1}{2} m a^2}$$

[2] [2] [6]

(b)



Using reference frame fixed to the Earth

direction of motion of truck ←

linear motion of CM: $F = -m\dot{v}$ ① [2]

ang. motion about CM: $Fa = I\dot{\omega}$ ② [2]

no-slip condition: $v - a\omega = -\frac{1}{2} g t$ ③ [2]

From ① and ② $I\dot{\omega} = -ma\dot{v}$

$a\dot{\omega} = -2\dot{v} \Rightarrow \boxed{a\omega = -2v}$ [2]

since $\omega = 0$ when $v = 0$

use boxed result in ③ to find:

$v + 2v = -\frac{1}{2} g t \Rightarrow \boxed{v = \frac{-1}{6} g t}$ ④ [2]

so, $a\omega = \frac{1}{3} g t$

$\boxed{a\theta = \frac{1}{6} g t^2}$ ⑤ since $\theta = 0$ at $t = 0$ [2]

↑ angle turned through

When leaves edge of truck: $a\theta = L = 5\,m$

From ⑤ $t^2 = \frac{6L}{g}$

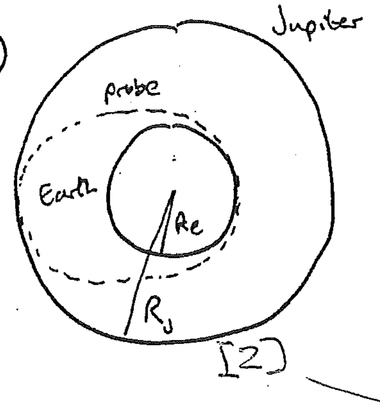
subst. in ④ $v = -\frac{1}{6} g \sqrt{\frac{6L}{g}} = -\sqrt{\frac{Lg}{6}}$

$= -\sqrt{\frac{5 \times 9.8}{6}}\ ms^{-1}$ [2]

or speed $2.86\ ms^{-1}$ in direction of motion of truck.

[14]

B2



Earth's orbit, radius R_e

Jupiter's orbit, radius $R_J = 5.2 R_e$

[16]

(i) For earth's orbit: [2]

$$\frac{v_e^2}{R_e} = \frac{GM}{R_e^2}$$

(ii) For the elliptical orbit of the probe.

Ang. mom: $L = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{L}{mr^2}$ .... ① [2]

Energy: $E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{GMm}{r^2}$ .... ② [2]

Equate energy at $r_{min} = R_e$ and $r_{max} = R_J$ using $\dot{r} = 0$ at those points:

$$\frac{1}{2}\frac{L^2}{mR_e^2} - \frac{GMm}{R_e} = \frac{1}{2}\frac{L^2}{mR_J^2} - \frac{GMm}{R_J} \quad [2]$$

$$\frac{L^2}{2m}\left(\frac{1}{R_e^2} - \frac{1}{R_J^2}\right) = GMm\left(\frac{1}{R_e} - \frac{1}{R_J}\right)$$

$$\frac{L^2}{2m} = \frac{GMm \; R_e R_J}{R_e + R_J} \quad [2]$$

At R_e, probe's speed is v and $L = mvR_e$

$$\Rightarrow v^2 = \frac{L^2}{m^2R_e^2} = 2\frac{GM}{R_e}\frac{R_J}{R_e+R_J}$$

$$= 2v_e^2\frac{R_J}{R_e+R_J} \quad [2]$$

$$\therefore \frac{v}{v_e} = \sqrt{\frac{2R_J}{R_e+R_J}} = \sqrt{\frac{2\times 5.2}{6.2}} = \boxed{1.3} \quad [2]$$

$1.1\sqrt{\frac{2}{1+1/5.2}} > \sqrt{2}$ so will escape

or note $\frac{v}{v_e} = \sqrt{\frac{2}{1+\frac{R_e}{R_j}}}$ so $\frac{v}{v_e} = \sqrt{2}$ if $R_j = \infty$

If initial speed is $1.1v$, then energy is,

$$E \overset{[2]}{=} \frac{1}{2}m(1.1)^2v^2 - \frac{GMm}{R_e} = \frac{GMm}{R_e}\left[(1.1)^2 \cdot \frac{R_J}{R_e+R_J} - 1\right] \quad [4]$$

$$= \frac{GMm}{R_e}(0.01) > 0$$ so probe will escape [2]

B3 (a) Starting from: $\ddot{\underline{x}} = \underline{g}^* - 2\underline{\omega}\times\dot{\underline{x}}$

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with $\underline{x} = \underline{a}$ and $\dot{\underline{x}} = \underline{v}$ at $t=0$ [2]

Integrate once, $\dot{\underline{x}} = \underline{g}^* t - 2\underline{\omega}\times\underline{x}$ + const.

use initial conditions: $\boxed{\dot{\underline{x}} = \underline{v} + \underline{g}^* t - 2\underline{\omega}\times(\underline{x}-\underline{a})}$ [2]

To order $\underline{\omega}$ can substitute $\underline{x}$ to order 1 in the corriolis term:

$$\underline{x} = \underline{v}t + \tfrac{1}{2}\underline{g}^* t^2 + \underline{a} + O(\underline{\omega}) \quad [2]$$

[10]

so: $$\dot{\underline{x}} = \underline{v} + \underline{g}^* t - 2\underline{\omega}\times(\underline{v}t + \tfrac{1}{2}\underline{g}^* t^2) + O(\omega^2)$$

Integrate: $$\underline{x} = \underline{v}t + \tfrac{1}{2}\underline{g}^* t^2 - 2\underline{\omega}\times(\underline{v}\tfrac{t^2}{2} + \tfrac{1}{6}\underline{g}^* t^3) + \text{const.} \quad [2]$$

Fix const by $\underline{x} = \underline{a}$ at $t=0$: const $= \underline{a}$

so $$\boxed{\underline{x} = \underline{a} + \underline{v}t + \tfrac{1}{2}\underline{g}^* t^2 - \tfrac{1}{3}\underline{\omega}\times\underline{g}^* t^3 - \underline{\omega}\times\underline{v}t^2} \quad [2]$$

(b) dropped from a tower. Choose upward vertical to be defined from $\underline{g}^*$.

So, $\underline{a} = h\hat{z}$, $\underline{v} = 0$, $\underline{g}^* = -g\hat{z}$ [2]

Let $\hat{x}$ be East, $\hat{y}$ North.

In latitude λ, $\underline{\omega} = \hat{z}\,\omega\sin\lambda + \hat{y}\,\omega\cos\lambda$

$$\underline{\omega}\times\underline{g}^* = -\omega g\cos\lambda\,\hat{x} \quad [2]$$

Hence in components:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} - \tfrac{1}{2}gt^2\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \tfrac{1}{3}\omega g\cos\lambda\, t^3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

[8]

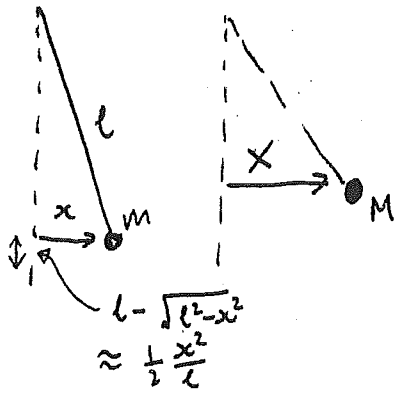
$z = 0$ when $t = \sqrt{\frac{2h}{g}}$, and then $x = \frac{1}{3}\omega g\cos\lambda.\left(\frac{2h}{g}\right)^{3/2}$ [2]

$$= \boxed{\frac{1}{3}\omega\left(\frac{8h^3}{g}\right)^{1/2}\cos\lambda} \quad [2]$$

c) Eastward deflection since x +ve. [2]

Normal mode: every part of system oscillates at same frequency [2]

B4



For small oscillations:

$$L = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} M \dot{X}^2 - \tfrac{1}{2} g \left(\frac{x^2 m}{l} + \frac{X^2 M}{l} \right) - \tfrac{1}{2} k (X - x)^2$$

so E–L eqns give:

$$m\ddot{x} = -\frac{mgx}{l} + k(X - x) \quad [2]$$

$$M\ddot{X} = -\frac{MgX}{l} - k(X - x) \quad [2]$$

Look for normal mode: $x = Ae^{i\omega t}$, $X = Be^{i\omega t}$

$$\begin{pmatrix} \frac{g}{l} + \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{M} & \frac{g}{l} + \frac{k}{M} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \omega^2 \begin{pmatrix} A \\ B \end{pmatrix} \quad [2]$$

[8]

Allowed frequencies: $\left(\frac{g}{l} - \omega^2 + \frac{k}{m} \right)\left(\frac{g}{l} - \omega^2 + \frac{k}{M} \right) - \frac{k^2}{mM} = 0$

$$\left(\frac{g}{l} - \omega^2 \right)\left(\frac{g}{l} - \omega^2 - \frac{k}{m} - \frac{k}{M} \right) = 0$$

$$\boxed{\omega^2 = \frac{g}{l} \equiv \omega_0^2 \quad \text{or} \quad \omega^2 = \frac{g}{l} + \frac{k}{m} + \frac{k}{M} \equiv \omega_1^2} \quad [2]$$

(i) when $\omega^2 = \frac{g}{l}$ find: $\begin{pmatrix} \frac{k}{m} & -\frac{k}{m} \\ -\frac{k}{M} & \frac{k}{M} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ mode $\boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ [2]

(as expected

[2]

(ii) when $\omega^2 = \frac{g}{l} + \frac{k}{m} + \frac{k}{M}$ find: $\begin{pmatrix} -\frac{k}{M} & -\frac{k}{m} \\ -\frac{k}{M} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$ mode: $\boxed{\begin{pmatrix} 1 \\ -\frac{m}{M} \end{pmatrix}}$

[4]

Motion starting from $\begin{pmatrix} x \\ X \end{pmatrix} = \begin{pmatrix} 0 \\ a \end{pmatrix}$, $\begin{pmatrix} \dot{x} \\ \dot{X} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

[6]

Write: $\begin{pmatrix} x \\ X \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos \omega_0 t + \beta \begin{pmatrix} 1 \\ -\frac{m}{M} \end{pmatrix} \cos \omega_1 t$ [2]

[2]

$\Rightarrow \alpha + \beta = 0$

$\alpha - \frac{\beta m}{M} = a$

[2]

$\Rightarrow \alpha = -\beta = \frac{Ma}{m+M}$, so solution is $\boxed{\begin{pmatrix} x \\ X \end{pmatrix} = \frac{aM}{m+M} \begin{pmatrix} \cos \omega_0 t - \cos \omega_1 t \\ \cos \omega_0 t + \frac{m}{M} \cos \omega_1 t \end{pmatrix}}$