PHYS2006 2011/2012

Sem 1 Exam

(A1) $\underline{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \underline{r}_i$; $M = \sum_{i=1}^{N} m_i$ [2]

where m_i are the masses and $\underline{r}_i$ the positions for the N particles. Thus

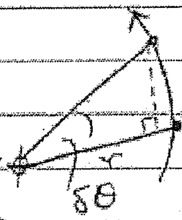
$\dot{\underline{R}} = \frac{1}{M}\sum_{i=1}^{N} m_i\dot{\underline{r}}_i = \frac{1}{M}\underline{P}$ or $\boxed{\underline{P} = M\dot{\underline{R}}}$ [2]

since the individual momenta are $m_i\dot{\underline{r}}_i$

(A2) For a force $\underline{F}$ applied at a point $\underline{r}$ the torque $\underline{\tau} = \underline{r}\times\underline{F}$ (1 for including const)

$\underline{\tau} = \frac{d}{dt}\underline{L}$ where $\underline{L}$ is angular momentum. (1) [2]

If the force is central then $\underline{F}\propto\underline{r}$, so $\underline{\tau} = 0$ (1) $\Rightarrow$ $\underline{L}$ = constant w.r.t. time ie. conserved (1) [2]

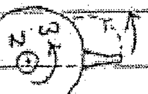
(A3) 

In time δt, the particle's radius sweeps out an area $\delta A = \frac{1}{2}r^2\delta\theta$ (1) therefore $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$ (1) [2]

(1) but $L = mr(r\dot{\theta}) = mr^2\dot{\theta}$ is a constant, so $\frac{dA}{dt}$ is a constant implying that equal (1) areas are swept out in equal times. This is Kepler's 2nd law. [2]

NB inertial

(A4) Looking down from the N pole, the ball is projected tangentially when viewed from an inertial frame. Gravitational force is central so angular momentum is conserved. (1) (1) [2]



Therefore as ball falls, it's angular velocity increases (1)

(A4) continued –
with the consequence that the ball gets slightly ahead of the tower as it falls. [2]

1

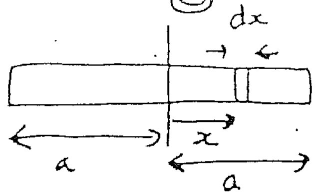
(A5) The Coriolis force is the term
$$-2m\,\underline{\omega} \times \underline{\dot{x}}$$
that appears in Newton's equation of motion for a particle when couched in the rotating frame of the Earth. Here $\underline{\omega}$ is the angular velocity of the Earth, and m and $\underline{\dot{x}}$ are the mass and (apparent) velocity of the particle. [2].

It thus supplies a force at right angles to the direction of motion. Foucault's pendulum is simply a pendulum attached to a stationary frame, typically very large so that it can swing unaided for a long time. Then it is found that the plane in which it swings precesses (clockwise from above) at an angular velocity of $\omega \sin\lambda$, λ being the latitude, as may be derived from the Coriolis force

For substantially correct [2] B

(B1)

(a)

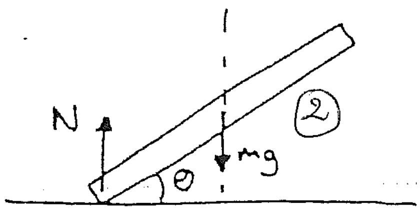


Let $\rho =$ ~~density~~ mass per unit length.

$$dI = \underset{(2)}{\rho\, dx\, x^2} \Rightarrow I = \int_{-a}^{a} \rho x^2 dx = \underset{(2)}{} \frac{2\rho a^3}{3} = \boxed{\frac{1}{3} m a^2}$$

[6]

(b) CM of rod descends in a vertical line, [2] one end remains in contact with table, but slides across it. Apply conservation of energy $\longrightarrow$



initial pe

$$mga = mga\sin\theta + \frac{1}{2}\cdot\frac{1}{3}ma^2\dot\theta^2 + \frac{1}{2}ma^2\cos^2\theta\,\dot\theta^2 \quad (2)$$

pe — rot. ke about CM — linear ke of cm.

[4]

$$\left(\frac{1}{6}ma^2 + \frac{1}{2}ma^2\cos^2\theta\right)\dot\theta^2 = mga(1-\sin\theta) \qquad [2]$$

when $\theta = 0$ find:

$$\dot\theta^2_{\theta=0} = \frac{ga}{\frac{1}{6}a^2 + \frac{1}{2}a^2} = \frac{3g}{2a} \quad (2)$$

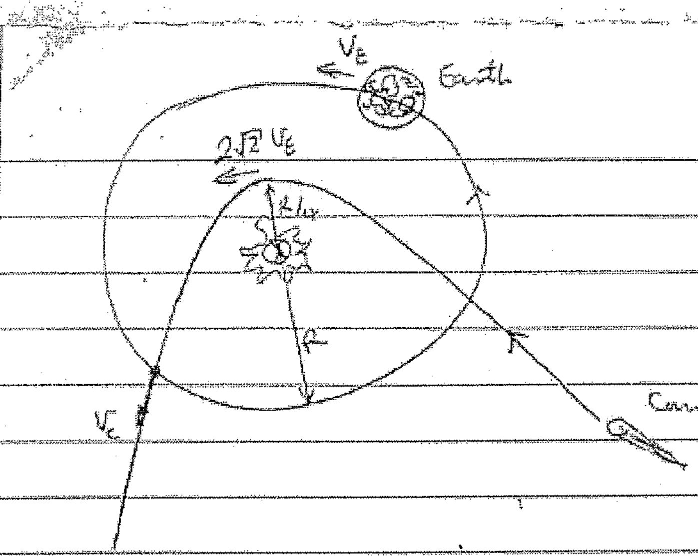
$$|\dot\theta|_{\theta=0} = \sqrt{\frac{3g}{2a}} \quad (2) \qquad [4]$$

First parts of (b): (1) CM of rod descends vertically

(2) Reaction force $\underline{N}$ does no work, since it is displaced perpendicular to its direction [1]

(3) Energy is conserved. [1]

B2



② for a decent diag and/or expl.

At closest approach comets velocity is ⊥ to its displacement from the sun therefore its angular momentum ①

$$L = m\left(\frac{R}{4}\right)\left(2\sqrt{2}\,V_E\right)$$ ① for formula for L (inc. quoted)

$$= \frac{1}{\sqrt{2}} m R V_E \qquad (*) \qquad [5]$$ ①

Its total energy $E = \frac{1}{2}m\left(2\sqrt{2}V_E\right)^2 - \frac{GMm}{R/4}$

$$= 4mV_E^2 - 4\frac{GMm}{R} \qquad (1)$$ ② 1 for correct statement of pot + KE

When it crosses Earths orbit with speed V_c, ② 1 for mention conserv. Energy

$$E = \frac{1}{2}mV_c^2 - \frac{GMm}{R} \qquad \text{(consvt}^n\text{ energy)} \quad (2)$$

But Earth's motion $\frac{V_E^2}{R} = \frac{GM}{R^2}$ (circular motion) (3) ② (one for mentioning correct idea of circular motion)

$$\therefore (1)\,\&\,(2) \Rightarrow \quad 4V_E^2 - 3\frac{GM}{R} = \frac{1}{2}V_c^2$$

$$\&\,(3) \Rightarrow \quad = V_E^2 \quad \Rightarrow \quad \underline{\underline{V_c = \sqrt{2}\,V_E}} \qquad [7]$$ ①

From (*) we have that the tangential component of velocity when comet crosses Earth's orbit so $V_t = V_E/\sqrt{2}$ ② (1 for correct idea)

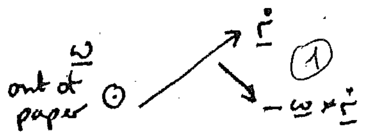
∴ we have [Comet: $\sqrt{2}\,V_E$; Earth; θ; $V_E/\sqrt{2}$] $\Rightarrow \cos\theta = \frac{1}{2}$ ② (1 for correct idea)

[5]

i.e. angle at crossing of $\underline{\underline{\theta = 60^\circ}}$ ①

(B3) ☐ Two inertial force terms in eqn. motion.

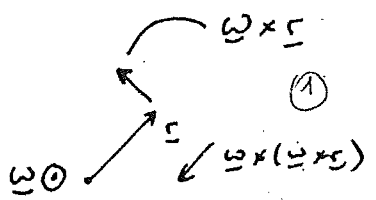
(i) Coriolis term $\quad -2m\underline{\omega}\times\dot{\underline{r}} \quad$ [2]



Direction of cross product shows that (2) bug feels a force perp to its velocity, directed towards the right.

magnitude: $\boxed{2m\omega u}$ [3]

(ii) centrifugal term. $\quad -m\underline{\omega}\times(\underline{\omega}\times\underline{r}) \quad$ [2]



$\underline{\omega}\times(\underline{\omega}\times\underline{r})$ points radially inwards, (2)

so $-\underline{\omega}\times(\underline{\omega}\times\underline{r})$ is radially outwards

magnitude $\boxed{m\omega^2 r}$ [3]

– normal "centrifugal force" term.

☐ For bug not to slip: consider horizontal part of eqn. of motion. [2]

$\ddot{\underline{r}} = 0$ in the rotating frame so, [2]

$$\underline{F} = 2m\underline{\omega}\times\dot{\underline{r}} + m\underline{\omega}\times(\underline{\omega}\times\underline{r})$$

[2]

o we clearly want $|\underline{F}| < \mu m g$ for no-slip (using vertical part of eqn. motion)

o Coriolis and centrifugal terms are orthogonal, so

$$\mu^2 g^2 < 4\omega^2 u^2 + \omega^4 r^2 \qquad [2]$$

So it can get to

$$\boxed{r = \frac{[\mu^2 g^2 - 4\omega^2 u^2]^{1/2}}{\omega^2}}$$

[2] before it slips

B2 continued –

Comet can escape if it has zero or +ve energy at ∞ distance from sun (= its kinetic energy there) ①

but energy will be

$$E = 4mV_E^2 - 4\frac{GMm}{R} \quad (\text{by } ①)$$

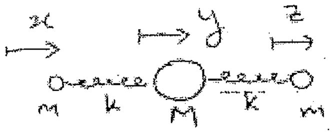
$$= 0 \quad (\text{by } ③)$$ ①

$\therefore$ Comet has just enough energy to escape ① [3]

(B4) • Normal mode: each part of the system oscillates with same frequency, but in general different amplitude and phase.

[1]B

[1]B

• 

[1]

$$N\underline{II} \Rightarrow \quad m\ddot{x} = -k(x-y) \qquad [1]$$
$$M\ddot{y} = k(x-y) - k(y-z) \qquad [1]$$
$$m\ddot{z} = k(y-z) \qquad [1]$$

Thus $$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = \begin{pmatrix} -k/m & k/m & 0 \\ k/M & -2k/M & k/M \\ 0 & k/m & -k/m \end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$ [2] (1^B for matrix, 1 for d...)

Look for normal mode: $\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}A\\ B\\ C\end{pmatrix} e^{i\omega t}$ [1]B

Require $$\begin{vmatrix} \omega^2 - k/m & k/m & 0 \\ k/M & \omega^2 - k/M & k/M \\ 0 & k/m & \omega^2 - k/m \end{vmatrix} = 0$$ (1^B for det, 1 for a...) [2]

$$\Rightarrow (\omega^2 - k/m)\{(\omega^2 - 2k/M)(\omega^2 - k/m) - k^2/mM\} - \tfrac{k}{m}\tfrac{k}{M}(\omega^2 - k/m) = 0 \quad [1]$$

$$\Rightarrow \omega^2(\omega^2 - k/m)(\omega^2 - k/m - 2k/M) = 0 \quad [1]$$

* • $\omega = 0$: find $A = B = C$ [1]* ⇒ mode of uniform translation of whole system at const. velocity [1] *

Also/or for intuitive derivation

• $\omega = \sqrt{\frac{k}{m}}$: $$\begin{pmatrix} 0 & k/m & 0 \\ k/M & k/m - 2k/M & k/M \\ 0 & k/m & 0 \end{pmatrix}\begin{pmatrix}A\\ B\\ C\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$ $B = 0 \quad A = -C$ [1] [1]

Central mass at rest. Outer masses oscillate out of phase

• $\omega = \left(\frac{k}{m} + \frac{2k}{M}\right)^{\frac{1}{2}}$: $$\begin{pmatrix} 2k/M & k/m & 0 \\ k/M & k/m & k/M \\ 0 & k/m & 2k/M \end{pmatrix}\begin{pmatrix}A\\ B\\ C\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$ $A = C$, $B = -\frac{2m}{M}C$ [1]

Outer two masses oscillate in phase; inner mass out of phase with outer masses, different amplitude. [1]