B = Bookwork i.e. lect notes, discussed in lects, Problem sheets etc.
(B) = Partly bookwork.

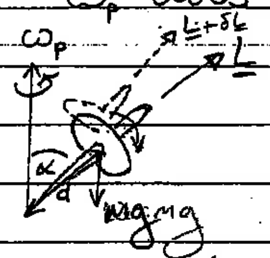
A1

Inside the sphere $g(r) = 0$ $[\frac{1}{2}]_B$

Outside $g(r) = -\frac{Gm\,\hat{r}}{r^2}$ $[\frac{1}{2}]_B$ $[\frac{1}{2}]_B$ $[\frac{1}{2}]_B$ $[\frac{1}{2}]_B$ $[\frac{1}{2}]_B$ $[\frac{1}{2}]_B$

where $G$ is Newton's gravitational constant and $\hat{r}$ is the unit vector $\underline{r}/r$ in the direction $\underline{r}$ $[\frac{1}{2}]_B$

A2 $\omega_p$ does not depend on $\alpha$ $[1]_B$



Torque $\tau = "\underline{r} \times \underline{F}"$
thus $\tau = dmg \sin\alpha$ $\Big\}$ $[1]_B$

$[1]_B$ for diag and/or def$^{ns}$ $d$

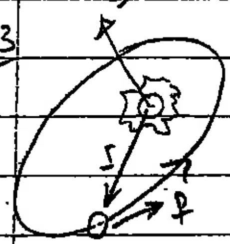
In a small time $\delta t$

$\delta L = |\delta \underline{L}| = L \sin\alpha\, \omega_p \delta t$

$\therefore \frac{dL}{dt} = \omega_p L \sin\alpha$

$\tau = \frac{dL}{dt} \Rightarrow \omega_p = \frac{dmg}{L}$ $\Big\}$ $[1]_B$

A3 $[\frac{1}{2}]_B$



with the Sun at the origin of coordinates, angular momentum $\underline{L} = \underline{r} \times \underline{p}$ $[1]_B$ for the planet is conserved $[1]_B$ This is because gravitational force is central and therefore the total torque $= \underline{r} \times \underline{F} = 0$ $[\frac{1}{2}]_B$ Consequently $\underline{L}$ forms a fixed direction in space and $\underline{r}$ and $\underline{p}$ are confined to the plane orthogonal to $\underline{L}$ $[1]_B$

A4 $\underline{R}$ is the position of the point in Southampton relative to the centre of the Earth, using axes which rotate with the Earth. [1]B

[1]B $\omega$ is the vector angular velocity of the Earth (and thus points north and has magnitude ... )

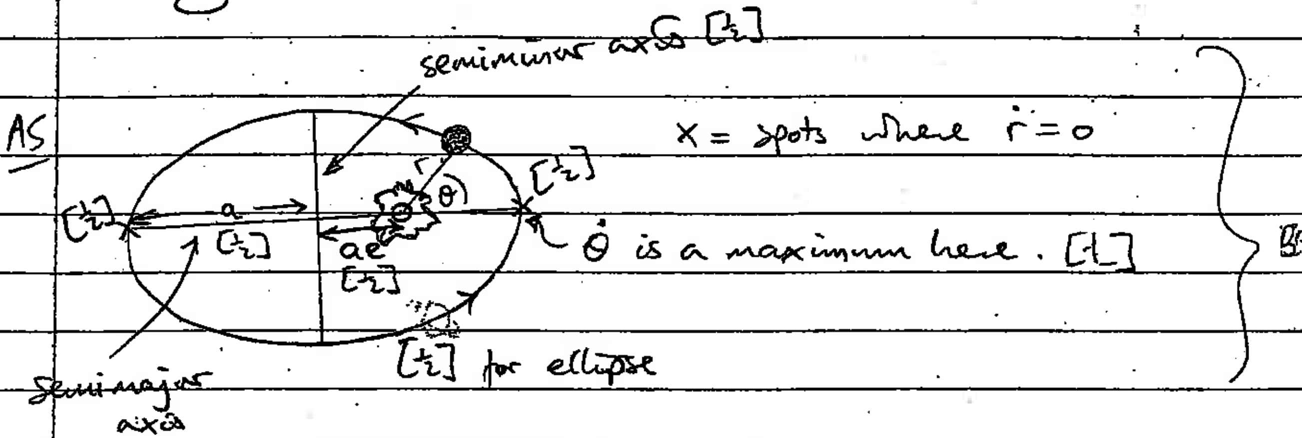
[1]B $-2\omega \times \dot{r}\, m$, where m is the mass of the ball, is the coriolis force. It is proportional to the balls velocity $\dot{r}$ and acts perpendicular to it.

For mostly complete answers

$m\omega\times(\omega\times R)$ is the centrifugal force due to the Earth spinning. In total $g\frac{R}{R} + \omega\times(\underline{\omega}\times \underline{R}) \equiv g^*$

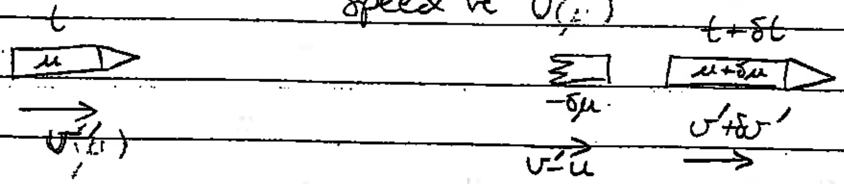
[1]B ~~gravity~~ an apparent gravity pointing slightly more towards the equator and with slightly smaller magnitude than g.

A5




B5

B1) Let the total mass of the rocket at time $t$ be $\mu$, & its speed be $v'(t)$ [1]$_B$




Conservation of momentum of isolated system [1]$_B$ $\Rightarrow$

$$\mu v' = (\mu+\delta\mu)(v'+\delta v') - \delta\mu\,(v'-u) \quad [2]_B$$

$$\Rightarrow \quad 0 = \mu\,\delta v' + \delta\mu\, u \quad [1]_B \quad (\text{to 1st order } [1]_B)$$

$\therefore$

Velocity when $m$ of fuel remains $\quad v = v_i - u\int_{C+M}^{C+m} \frac{d\mu}{\mu} = u\ln\left(\frac{C+M}{C+m}\right)$ [1]$_B$

initial velocity $= 0$ [1]$_B$]

Initial velocity for 2nd stage $= v$ [1]$_B$ & initial mass $= C+m$ [1]$_B$

Final " " " " $= v_f$ [1]$_B$ & final mass $= C$ [1]$_B$

marks given if clearly implication

opposite thrust [1]$_B$

$$\therefore \quad v_f = v + u\ln\left(\frac{C+m}{C}\right) \quad [1]$$

$$= u\ln\left(\frac{C+M}{C+m}\right) + u\ln\left(\frac{C}{C+m}\right) \quad [1]_B$$

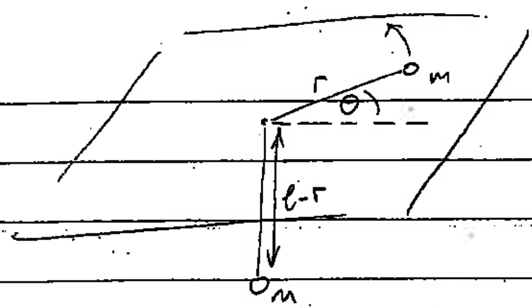
$$= u\ln\left(\frac{C(C+M)}{(C+m)^2}\right)$$

$$v_f = 0 \Rightarrow \frac{C(C+M)}{(C+m)^2} = 1 \quad [1]$$

i.e. $\quad C+m = \sqrt{C(C+M)}$ [1] [1] (− sign $\Rightarrow$ −ve $m$! so discard)

or $\quad m = \sqrt{C(C+M)} - C$ [2]

B2



A novel question that adapts what the students should know from ~~planetary motion~~ [1]

Let the mass of the particles be $m$.

$$K.E. \text{ of particle on the table} = \overbrace{\tfrac{1}{2}m\dot{r}^2}^{[2](1)} + \overbrace{\tfrac{1}{2}mr^2\dot{\theta}^2}^{[2](1)}$$

Angular momentum of particle on the table $= mr^2\dot{\theta}$ [2]

Conserved:

[1] Angular momentum $= mr^2\dot{\theta}$ [1] —①

[1] Total energy (up to additive constant) $= \underset{[1]}{m\dot{r}^2} + \tfrac{1}{2}mr^2\dot{\theta}^2 + \underbrace{mrg}_{[2]}$ —②

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Initial angular momentum $= m\frac{\ell}{2}v$ —③ [1]

Initial total energy $= \frac{1}{2}mv^2 + m\frac{\ell}{2}g$ —④ [2]

Equating ① & ③ $\Rightarrow$ $r^2\dot{\theta} = \frac{1}{2}\ell v$ [1]

& substituting for $\dot{\theta}$ in ② equated to ④ $\Rightarrow$ [2]

$$\dot{r}^2 + \frac{1}{8}\frac{\ell^2v^2}{r^2} + rg = \frac{v^2}{2} + \frac{\ell}{2}g \qquad [1]$$

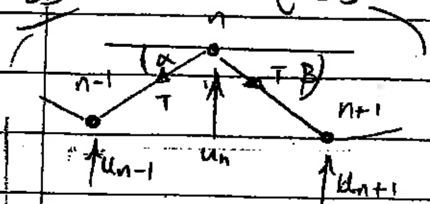
If the hanging particle is to reach the hole its kinetic energy must still be non-negative [1]

i.e. $0 \leq \dot{r}^2_{final} = \frac{v^2}{2} + \frac{\ell}{2}g - \left(\frac{1}{8}v^2 + \ell g\right)$ [1]

$$= \frac{3v^2}{8} - \frac{\ell}{2}g$$

i.e. $\underline{\underline{v^2 \geq \frac{4}{3}\ell g}}$

BB [1]$_B$ [1]$_B$




$$\text{NII}: \; m\ddot{u}_n = -T\sin\alpha - T\sin\beta \qquad [1]_B$$

$$\sin\alpha \simeq \frac{(u_n - u_{n-1})}{a} \qquad \sin\beta \simeq \frac{(u_n - u_{n+1})}{a} \qquad [1]_B$$

for small displacements. [1]$_B$

$$\text{so} \quad m\ddot{u}_n = \frac{T}{a}(u_{n-1} - 2u_n + u_{n+1}) \qquad [2]_B$$

Substitute normal mode $u_n = Ae^{i\omega t}e^{in\theta}$

[1]$_B$ [1]$_B$ [1]$_B$

$$\Rightarrow \; -\omega^2 m = \frac{T}{a}\left(e^{-i\theta} - 2 + e^{i\theta}\right) = \frac{2T}{a}(\cos\theta - 1)$$

$$\text{i.e.} \quad \underline{\underline{\omega^2 = \frac{2T}{ma}(1-\cos\theta)}}$$

$u_0 = 0$ implied [1] (B)

$u_5 = h\cos(\omega t)$ (given)

Since the interactions between beads are nearest neighbour only, it's enough to find linear combination of normal modes of ∞ system satisfying the above. [1] (A) [1]

[1]

$u_5 \Rightarrow \cos(\omega t)$ & $u_0 \Rightarrow 0 \Rightarrow u_n = A\cos\omega t\sin(n\theta)$ [1]

Dispersion relⁿ $\Rightarrow \cos\theta = 1 - \frac{ma\omega^2}{2T} = 1 - \frac{10^{-3}\times 2\times 10^{-2}\times 100}{2\times 2\times 10^{-3}}$

$$= \tfrac{1}{2} \qquad [1]$$

And $u_5 \Rightarrow A = h/\sin(5\theta)$ [1]

∴ without loss of generality, take solⁿ $\theta = \frac{\pi}{3}$ & then [1]

[1]

$$u_2 = \frac{h}{\sin\left(\frac{5\pi}{3}\right)}\cos\omega t\,\sin\left(\frac{2\pi}{3}\right) = -h\cos(\omega t) \qquad [1]$$

$$= \underline{\underline{-0.1\,\text{cm}\cos(10t)}}$$

B4

(a) Consider MoI about 3 orthogonal axes with orig in ? centre.

$$I_x = \int_{vol} (y^2+z^2)\, dm \quad [1]_B$$

$$I_y = \int (x^2+z^2)\, dm \quad [1]$$

$$I_z = \int (x^2+y^2)\, dm \quad [1]$$

$[3]_B$

Symmetry $\Rightarrow$ $I = I_x = I_y = I_z$ $[1]_B$

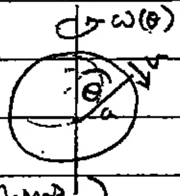
So $3I = I_x + I_y + I_z = 2\int (x^2+y^2+z^2)\, dm$ $[1]_B$

$$= 2\, 4\pi\rho \int_0^a r^4\, dr \quad [2]_B$$

$$= \frac{8\pi\rho a^5}{5} \quad [1]_B$$

Now $m = \frac{4\pi}{3}\rho a^3$ $[1]_B$ $\therefore$ $\underline{\underline{I = \frac{2}{5} m a^2}}$

(b)



[1] (and/or diagram)

By conservation of angular momentum [1]

$$\underbrace{I\omega}_{[1]} = \Big(\underbrace{I}_{[1]} + \underbrace{\tfrac{2}{5} m a^2 \sin^2\theta}_{[1]}\Big)\, \omega(\theta)$$

$$\Rightarrow \quad \underline{\underline{\omega(\theta) = \frac{\omega}{1+\sin^2\theta}}} \quad [1]$$

Insect walks with constant speed reaching $\theta = \pi$ [1] at time $t = T$. [1]

Thus $\theta(t) = \pi\, t/T$ [1]

Total angle turned by sphere $= \underbrace{\int_0^T \omega(t)\, dt}_{[1]} = \frac{T}{\pi}\int_0^\pi \omega(\theta)\, d\theta = \frac{T\omega}{\pi}\frac{\pi}{\sqrt{2}} = \underbrace{\frac{\omega T}{\sqrt{2}}}_{[1]}$