SEMESTER 2 EXAMINATION 2016-2017

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books.

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct 'Word to Word' translation dictionary AND it contains no notes, additions or annotations.

16 page examination paper.

Section A

A1. Show that, if a fixed-length vector $\mathbf{A}$ rotates with angular velocity ω about an axis defined by the vector $\hat{\boldsymbol{\omega}}$, and we define $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{\omega}}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}.$$ [4]

From a suitable diagram, we see that the infinitessimal change $\mathrm{d}\mathbf{A}$ resulting from rotation of $\mathbf{A}$ through an infinitessimal angle $\mathrm{d}\varphi$ about $\hat{\boldsymbol{\omega}}$ will be

$$\mathrm{d}\mathbf{A} = \hat{\boldsymbol{\omega}} \times \mathbf{A}\mathrm{d}\varphi.$$ *[2]*

Dividing by an infintessimal timestep $\mathrm{d}t$ and noting that the angular velocity $\omega \equiv \mathrm{d}\varphi/\mathrm{d}t$,

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \hat{\boldsymbol{\omega}} \times \mathbf{A}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega\hat{\boldsymbol{\omega}} \times \mathbf{A} = \boldsymbol{\omega} \times \mathbf{A}.$$ *[2]*

A2. State Kepler's laws and outline the physical assumptions upon which they are based. [4]

Kepler's laws of planetary motion are

(a). The orbit of a planet is an ellipse with the Sun at one of its foci *[1]*

(b). The line from the Sun to the planet sweeps out equal areas in equal intervals of time *[1]*

(c). The square of the orbital period is proportional to the cube of the semi-major axis of its orbit *[1]*

Kepler's laws assume conservation of energy, conservation of angular momentum, and non-relativistic, Euclidean space; it may also/alternatively be mentioned that they assume gravity to be a central force obeying the inverse square law, and that no external forces/torques act. In the stated form, $M \gg m$ and any tidal forces or effects of non-sphericity are neglected. *[1]*

A3. Show, either by adapting Gauss's law or by direct integration, that the gravitational attraction exerted at a point $\mathbf{r}$ by a spherically symmetrical mass distribution is the same as that exerted by a point mass, positioned at the centre of spherical symmetry $\mathbf{R}$, whose mass is equal to that of the part of the distribution that lies within a distance $|\mathbf{r} - \mathbf{R}|$ of $\mathbf{R}$. [4]

Since the gravitational and electrostatic forces both obey an inverse-square law, both follow a divergence theorem [1], i.e.,

$$\oiint_A \mathbf{g}\cdot \mathrm{d}A = -4\pi G \iiint_V \mathrm{d}m$$

where $\mathbf{g}$ is the gravitational acceleration, G the gravitational constant, $\mathrm{d}m$ an element of mass, and the integrals are over the surface and volume of a given region. For a spherically-symmetrical mass distribution, and concentric spherical integration region whose surface includes the measurement point, the value of $\mathbf{g}\cdot\mathrm{d}A$ will be the same at all points on the sphere, while rotational symmetry demands that azimuthal components of $\mathbf{g}$ be zero [1]; a local value of $\mathbf{g}$ may hence be obtained [1]:

$$|\mathbf{g}| = \left|\frac{1}{4\pi}\oiint_A \mathbf{g}\cdot\mathrm{d}A\right| = \left|\frac{1}{4\pi}(-4\pi G)\iiint_V \mathrm{d}m\right| = \iiint_V G\mathrm{d}m$$

The gravitational acceleration hence depends upon the total mass contained within the spherical region, and is not affected by mass that lies beyond it. Since the radial mass distribution within this region does not matter, the acceleration is the same as if the mass were concentrated at the centre of the sphere [1]. *[4]*

[To obtain the result by direct integration, we consider a thin spherical shell of radius a, mass per unit area ρ and total mass $m = 4\pi\rho a^2$, concentric with the mass distribution. We break the shell into thin rings of width $a\,\mathrm{d}\vartheta$ defined by the angle ϑ to the line from $\mathbf{R}$ to $\mathbf{r}$, and of mass $\mathrm{d}m = \rho 2\pi a\sin\vartheta a\mathrm{d}\vartheta = (m/2)\sin\vartheta\mathrm{d}\vartheta$. The contribution of the annulus to the gravitational potential Φ at $\mathbf{r}$ is [1]

$$\mathrm{d}\Phi = -\frac{G\,\mathrm{d}m}{R} = -\frac{Gm}{2}\frac{\sin\vartheta\mathrm{d}\vartheta}{R}$$

where R is the distance from each point on the ring to $\mathbf{r}$. Using the cosine rule,

$$R^2 = r^2 + a^2 - 2ar\cos\vartheta$$

where $r \equiv |\mathbf{R}-\mathbf{r}|$, we find that $\sin\vartheta\mathrm{d}\vartheta/R = \mathrm{d}R/(ar)$ [1], and the contribution to the gravitational potential from the entire shell is hence [1]

$$\Phi(r) = -\frac{Gm}{2ar}\int_{|r-a|}^{r+a}\mathrm{d}R = \begin{cases} -Gm/r & \text{for } r \geq a \\ -GM/a & \text{for } r < a \end{cases}.$$

Since for $r < a$ the potential is constant, there is no gravitational force from a shell that encloses the measurement point. The force is independent of a, and hence the same as if the mass were at the centre of the distribution [1].]

A4. Show, using the *perpendicular axis theorem* or otherwise, that the moment of inertia I of a square plate, about a perpendicular axis through its centre, is given by

$$I = \frac{MA}{6}$$

where M is the mass of the plate and A its area. [4]

The moment of inertia may be found by determining the moment of inertia about an axis of symmetry within the plane of the plate and applying the perpendicular axis theorem, i.e. [one mark per line], *[3]*

$$\begin{aligned} I_x = I_y &= \int_{-a/2}^{a/2} x^2 \rho a \mathrm{d}x \\ &= \rho a \left[\frac{x^3}{3}\right]_{-a/2}^{a/2} = \frac{\rho a}{3}\left[\left(\frac{a}{2}\right)^3 - \left(-\frac{a}{2}\right)^3\right] = \frac{\rho a^4}{12} \\ &= \frac{\rho a^2 a^2}{12} = \frac{MA}{12}. \end{aligned}$$

where in the last line we have used that the plate area is $A = a^2$ and mass is $M = \rho a^2$. Hence [1]

$$I_z = I_x + I_y = \frac{MA}{6}.$$ *[1]*

Alternatively, by double integration, the moment of inertia I of a square plate of side a and mass ρ per unit area will be [1 mark per line]

$$\begin{aligned} I &= \int_{-a/2}^{a/2}\int_{-a/2}^{a/2} \rho\left(x^2 + y^2\right) \mathrm{d}x\,\mathrm{d}y \\ &= \rho \int_{-a/2}^{a/2} \left[\frac{2}{3}\left(\frac{a}{2}\right)^3 + y^2 a\right] \mathrm{d}y \\ &= \rho\left[\frac{2}{3}\left(\frac{a}{2}\right)^3 a + \frac{2}{3}\left(\frac{a}{2}\right)^3 a\right] = \frac{\rho a^4}{6}. \end{aligned}$$

Since the plate area and mass will be $A = a^2$ and $M = \rho a^2$, we again find

$$I = \frac{MA}{6}.$$

A5. At a shooting range in central Australia (27° S), a rifle bullet is fired with an initial speed (*muzzle velocity*) v horizontally towards the west. Explain how, and in which direction, it is deflected as a result of the Earth's rotation. [4]

In the frame of the shooting range, the bullet experiences centrifugal and Coriolis forces, which reflect that the freely-moving bullet will continue in a straight line as the Earth rotates beneath it. *[1]*

Relative to the bullet, the shooting range will move along a circle perpendicular to the polar axis. Since the Earth rotates from west to east, the shooting range will rise towards a westward-moving bullet, which will hence appear to move downwards. *[1]*

The downward direction perpendicular to the polar axis will, for a point in the southern hemisphere, be angled towards the south by an angle equal to the latitude of the point. *[1]*

The bullet will hence be deflected downwards and to the left (south), at an angle of 27° to the vertical. *[1]*

Section B

B1. A bricklayer wishes to clear some left-over bricks from the top of a building, and rigs up a simple hoist by attaching an empty barrel of mass $m_1 = 10$ kg to a line that passes over a pulley at the top. He hoists the barrel to the top of the building, a height $h = 25$ m above the ground, and secures the line at the bottom. He then fills the barrel with bricks of total mass $m_2 = 90$ kg.

(a) Calculate (i) the tension in the line, and (ii) the load borne by the pulley. [4]

(i) Since the brick-bearing barrel, of total mass $(m_1 + m_2)$ is stationary, its weight must be balanced by the line tension T, so that

$$T = (m_1 + m_2)g \qquad (= 100g = 980\,\mathrm{N})\,.$$ *[2]*

(ii) Two parts of the line, each carrying a tension T, pull down on the pulley, so the load L that it bears will be

$$L = 2T = 2(m_1 + m_2)g \qquad (= 200g = 1960\,\mathrm{N})\,.$$ *[2]*

The bricklayer, of mass $m_3 = 80$ kg, then goes to the bottom and casts off the line. Unfortunately the barrel starts moving down, jerking him off the ground.

He decides to hang on.

(b) (i) Show that the acceleration a experienced by the barrel will be

$$a = g\,[1 - 2m_3/\,(m_1 + m_2 + m_3)]\,,$$

and hence (ii) calculate the load now borne by the pulley. [7]

(i) Writing y as the distances of the barrel and bricklayer from their starting points, Newton's second law gives *[2]*

$$\begin{aligned}(m_1 + m_2)\ddot{y} &= (m_1 + m_2)g - T \\ m_3\ddot{y} &= T - m_3 g.\end{aligned}$$

Adding these expressions to eliminate the tension T, we obtain

$$a \equiv \ddot{y} = g\frac{m_1 + m_2 - m_3}{m_1 + m_2 + m_3} = g\left[1 - \frac{2m_3}{m_1 + m_2 + m_3}\right] \qquad \left(= g/9 = 1.09\,\mathrm{m\,s^{-1}}\right).$$ *[2]*

(ii) Inserting this into our application of Newton's second law gives the tension *[2]*

$$\begin{aligned}T &= m_3\,(\ddot{y} + g) \\ &= m_3 g\left[2 - \frac{2m_3}{m_1 + m_2 + m_3}\right] \\ &= 2g\frac{(m_1 + m_2)\,m_3}{m_1 + m_2 + m_3}\end{aligned}$$

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and hence the load on the pulley will be

$$L = 2T = 4g\frac{(m_1 + m_2)\, m_3}{m_1 + m_2 + m_3} \qquad (= (1600/9)g = 1742\,\mathrm{N})\,.$$ *[1]*

Halfway up, the bricklayer meets the barrel coming down.

(c) Neglecting friction and air resistance, calculate the relative speed with which the barrel strikes the bricklayer. [3]

Under constant acceleration a *from rest, the speed* v *after travelling a distance* $y = h/2$ *will be given by*

$$v^2 = 2ay = ah$$ *[1]*

hence the relative speed will be

$$2v = 2\sqrt{2gy\left[1 - \frac{2m_3}{m_1 + m_2 + m_3}\right]} = 2\sqrt{gh\left[1 - \frac{2m_3}{m_1 + m_2 + m_3}\right]} \qquad \left(= 2\sqrt{25g/9} = 10.4\,\mathrm{m\,s^{-1}}\right).$$

[2]

The bricklayer continues to be pulled to the top, banging his head against the pulley beam. The barrel hits the ground and bursts at its bottom, allowing all the bricks to spill out, leaving just the barrel attached to the line.

(d) Assuming that the barrel was slowed to a halt when it struck the bricklayer, find the kinetic energy of the brick-laden barrel just before it hits the ground. [2]

The barrel falls the same distance as in (c), under the same acceleration, so its speed just before hitting the ground will be as above; its kinetic energy E *will hence be* *[2]*

$$\begin{aligned} E &= \frac{1}{2}(m_1 + m_2)\, v^2 \\ &= \frac{gh}{2}(m_1 + m_2)\left[1 - \frac{2m_3}{m_1 + m_2 + m_3}\right] \qquad (= 1250g/9 = 1361\,\mathrm{J})\,. \end{aligned}$$

The bricklayer is now heavier than the barrel, and so starts down again at high speed. Halfway down, he meets the barrel coming up.

(e) Calculate the relative speed of the barrel when it strikes the bricklayer. [2]

We may use the same expression as in (c), but set the barrel contents m_2 to zero and reverse the acceleration direction, giving

$$2v = 2\sqrt{-gh\left[1 - \frac{2m_3}{m_1 + m_3}\right]} \qquad \left(= 2\sqrt{175g/9} = 27.6\,\mathrm{m\,s^{-1}}\right). \qquad [2]$$

The bricklayer continues to fall, and lands uncomfortably on the bricks. At this point, he loses his presence of mind, and lets go of the line.

(f) Calculate the kinetic energy of the empty barrel when it lands on the bricklayer. [2]

Since, in the absence of friction and air resistance, energy is conserved, the kinetic energy E at the ground will equal the potential energy with which the barrel started its descent

$$E = m_1 gh \qquad (= 250g = 2450\,\mathrm{J})\,. \qquad [2]$$

[Gerard Hoffnung's original telling: `http://monologues.co.uk/004/Bricklayers_Story.htm`*]*

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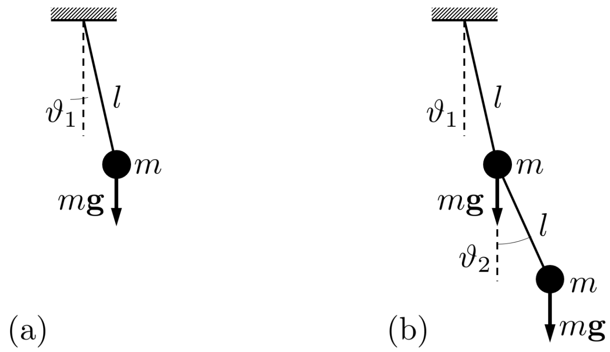
B2. (a) Explain what is meant by (i) *simple harmonic motion* and (ii) the *normal mode* of an oscillating system. [4]

(i) Simple harmonic motion is that of a single body when subject to a restoring force that is proportional to its displacement, so that the displacement varies sinusoidally with time . *[2]*

(ii) A normal mode is a motion in which all parts of the system oscillate with the same single frequency and (therefore) with a fixed phase relationship between each other. *[2]*



(b) A simple pendulum of length l has a bob of mass m, subject to gravitational acceleration g. The position of the pendulum bob is described by the angle ϑ_1 between the pendulum and the vertical, as shown in figure (a) above. Show, explaining any approximations made, that for small angles $|\vartheta_1| \ll 1$, the motion will be governed by the equation

$$\frac{\mathrm{d}^2\vartheta_1}{\mathrm{d}t^2} = -\frac{g}{l}\vartheta_1. \qquad [\,4\,]$$

If $\mathbf{W}_1$ is the tension in the pendulum, the force on the bob will be $m\mathbf{g} + \mathbf{W}_1$ [1]. Resolving and assuming no motion perpendicular to the pendulum arc, this gives $W_1 = mg\cos\vartheta_1$. Along the arc, the component $mg\sin\vartheta_1$ [1] will cause the bob to accelerate. If $x_1 \equiv l\vartheta_1$ is the position along the arc, we hence find

$$m\frac{\mathrm{d}^2x_1}{\mathrm{d}t^2} = m_1 l\frac{\mathrm{d}^2\vartheta_1}{\mathrm{d}t^2} = -mg\sin\vartheta_1 \approx -mg\vartheta_1,$$

where the approximation [1] is valid for small angles, and hence [1]

$$\frac{\mathrm{d}^2\vartheta_1}{\mathrm{d}t^2} = -\frac{g}{l}\vartheta_1. \qquad [\,4\,]$$

(c) A second pendulum, also of length l and with a bob of mass m, is suspended from the bob of the first pendulum, and its position described by its angle ϑ_2 to the vertical, as shown in figure (b) above. Show, again for small angles $|\vartheta_{1,2}| \ll 1$, that

$$\frac{\mathrm{d}^2\vartheta_1}{\mathrm{d}t^2} = \frac{g}{l}\left(\vartheta_2 - 2\vartheta_1\right). \qquad [3]$$

The tension W_2 in the second pendulum will, for small angles, be approximately mg [1]. This acts upon the first bob to give a component $mg\sin(\vartheta_2 - \vartheta_1) \approx mg(\vartheta_2 - \vartheta_1)$ [1] along the pendulum arc, where the approximation is valid for small angles. The acceleration of the bob is hence modified to [1]

$$\frac{\mathrm{d}^2\vartheta_1}{\mathrm{d}t^2} = -\frac{g}{l}\vartheta_1 + \frac{g}{l}\left(\vartheta_2 - \vartheta_1\right) = \frac{g}{l}\left(\vartheta_2 - 2\vartheta_1\right). \qquad [3]$$

(d) Show that, for small angles $|\vartheta_{1,2}| \ll 1$, the horizontal displacement of the second bob is approximately $l\left(\vartheta_1 + \vartheta_2\right)$ and hence show that

$$\frac{\mathrm{d}^2\vartheta_1}{\mathrm{d}t^2} + \frac{\mathrm{d}^2\vartheta_2}{\mathrm{d}t^2} = -\frac{g}{l}\vartheta_2. \qquad [3]$$

The horizontal displacement will be $l\sin\vartheta_1 + l\sin\vartheta_2$. For small angles $\vartheta_{1,2}$, this is approximately $l\vartheta_1 + l\vartheta_2$. *[1]*

As in part (b), the restoring force will have a component $mg\sin\vartheta_2 \approx mg\vartheta_2$ perpendicular to the line of the second pendulum [1]. (This is not necessarily perpendicular to the path of the second bob, but variations along the line of the bob will be accounted for by changes in the tension within the pendulum.) Hence, from Newton's second law,

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(l\vartheta_1 + l\vartheta_2\right) = -mg\vartheta_2$$

which, after cancelling the common factor m, gives the requested result [1]. *[2]*

(e) Show that the motion of the double pendulum is therefore governed by the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix}\vartheta_1\\ \vartheta_2\end{pmatrix} = \begin{pmatrix}-\frac{2g}{l} & \frac{g}{l}\\ \frac{2g}{l} & -\frac{2g}{l}\end{pmatrix}\begin{pmatrix}\vartheta_1\\ \vartheta_2\end{pmatrix}. \qquad [2]$$

The expressions of parts (c) and (d) are combined to give an expression for the acceleration of the second bob alone [1]

$$\frac{\mathrm{d}^2\vartheta_2}{\mathrm{d}t^2} = -\frac{g}{l}\vartheta_2 - \frac{\mathrm{d}^2\vartheta_1}{\mathrm{d}t^2}$$

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$$= -\frac{g}{l}\vartheta_2 - \frac{g}{l}(\vartheta_2 - 2\vartheta_1)$$
$$= 2\frac{g}{l}(\vartheta_1 - \vartheta_2).$$

This, together with the expression of part (c), are then written in matrix form as [1]

$$\frac{d^2}{dt^2}\begin{pmatrix}\vartheta_1\\ \vartheta_2\end{pmatrix} = \begin{pmatrix}-\frac{2g}{l} & \frac{g}{l}\\ \frac{2g}{l} & -\frac{2g}{l}\end{pmatrix}\begin{pmatrix}\vartheta_1\\ \vartheta_2\end{pmatrix}.$$ *[2]*

(f) Hence derive the frequencies of the modes of the double pendulum. [4]

For a normal mode, we may write [1]

$$\begin{pmatrix}\vartheta_1\\ \vartheta_2\end{pmatrix} = e^{i\omega t}\begin{pmatrix}a_1\\ a_2\end{pmatrix}$$

which, upon substitution into the expression of part (e), gives [1]

$$-\omega^2\begin{pmatrix}a_1\\ a_2\end{pmatrix} = \begin{pmatrix}-\frac{2g}{l} & \frac{g}{l}\\ \frac{2g}{l} & -\frac{2g}{l}\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}$$

and hence

$$\begin{pmatrix}\omega^2 - \frac{2g}{l} & \frac{g}{l}\\ \frac{2g}{l} & \omega^2 - \frac{2g}{l}\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix} = 0.$$

This requires that the determinant of the matrix above be zero, and hence [1]

$$\left(\omega^2 - \frac{2g}{l}\right)\left(\omega^2 - \frac{2g}{l}\right) - \frac{g}{l}\frac{2g}{l} = 0$$

which yields the quadratic equation [0.5]

$$\left(\omega^2\right)^2 - \frac{4g}{l}\omega^2 + \frac{2g^2}{l^2} = 0$$

to which the solutions are

$$\omega^2 = \frac{\frac{4g}{l} \pm \sqrt{\left[\frac{4g}{l}\right]^2 - 4\frac{2g^2}{l^2}}}{2} = \frac{g}{l}\left(2 \pm \sqrt{2}\right)$$

and hence [0.5]

$$\omega = \sqrt{\frac{g}{l}\left(2 \pm \sqrt{2}\right)}.$$ *[4]*

B3. A comet of mass m moves in the gravitational field of a star of mass M, and its position is described by its polar coordinates (r, ϑ) relative to the star. The gravitational potential is given by $\mathcal{V}(r) = GMm/r$. Assume that $M \gg m$.

(a) Show that the angular momentum of the comet about the star will be $L = mr^2\dot{\vartheta}$, where $\dot{\vartheta}$ signifies $\mathrm{d}\vartheta/\mathrm{d}t$, the rate of change of ϑ with time. [2]

The tangential (azimuthal) component of the velocity will be $r\dot{\vartheta}$ [1], so the angular momentum will be $L = mr\left(r\dot{\vartheta}\right) = mr^2\dot{\vartheta}$ [1]. *[2]*

(b) Show that the comet's total energy $\mathcal{E}$ may be written as

$$\mathcal{E} = \frac{m}{2}\dot{r}^2 + \left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right) \equiv \frac{m}{2}\dot{r}^2 + \mathcal{U}(r),$$

where $\dot{r} \equiv \mathrm{d}r/\mathrm{d}t$ and $\mathcal{U}(r)$ is the effective potential in which the comet's radial motion occurs. [4]

The kinetic energy may be written as [1]

$$\mathcal{T} = \frac{1}{2}mv^2 = \frac{m}{2}\left[\dot{r}^2 + \left(r\dot{\vartheta}\right)^2\right].$$

The second term may be written in terms of the conserved angular momentum L, eliminating $\dot{\vartheta}$ and giving [1]

$$\mathcal{T} = \frac{1}{2}mv^2 = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2}.$$

The total energy is the sum of the kinetic and potential energies [1]

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + \mathcal{V}(r) = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$

The last two terms depend only upon the radial coordinate r and may hence be combined into an effective potential [1]

$$\mathcal{E} = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \equiv \frac{m}{2}\dot{r}^2 + U(r),$$

where

$$\mathcal{U}(r) \equiv \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$ *[4]*

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(c) Assuming that the comet follows an elliptical orbit with the star at one focus, show from these results that the length $2a$ of the ellipse's major axis will be

$$2a = \frac{GMm}{-\mathcal{E}}.$$ [4]

The equation of motion of the comet is

$$\mathcal{E} = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

where the total energy $\mathcal{E}$ and angular momentum L are conserved. At the aphelion and perihelion, $\dot{r} = 0$, so [1]

$$\mathcal{E} - \frac{L^2}{2mr^2} + \frac{GMm}{r} = 0,$$

which yields the quadratic equation [1]

$$\mathcal{E}r^2 + GMmr - \frac{L^2}{2m} = 0.$$

This has solutions [1]

$$r = \frac{-GMm \pm \sqrt{(GMm)^2 - 4\mathcal{E}\frac{L^2}{2m}}}{2\mathcal{E}}$$

The length of the major axis of the ellipse will be the sum of these two solutions, i.e. [1],

$$2a = \frac{GMm}{-\mathcal{E}}.$$ *[4]*

(d) By differentiating the total energy with respect to the time t, derive the equation of radial motion of the comet,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{L^2}{m^2 r^3} - \frac{GM}{r^2}.$$ [2]

Differentiating the expression given in (b), where the total energy $\mathcal{E}$ and angular momentum L are conserved, we find [1]

$$0 = \frac{m}{2}2\dot{r}\ddot{r} - \frac{L^2}{mr^3}\dot{r} + \frac{GMm}{r^2}\dot{r}$$

whence, dividing through by $\dot{r}$ and rearranging, we obtain [1]

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{L^2}{m^2 r^3} - \frac{GM}{r^2}.$$ *[2]*

(e) By writing $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \dot{\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta} \equiv \frac{L}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\vartheta}$ and making the substitution $r \equiv 1/u$, show that the equation of motion may be rewritten as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\vartheta^2} = -u + \frac{GMm^2}{L^2}.$$ [4]

Making the substitutions suggested, [1]

$$\frac{Lu^2}{m}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\frac{L}{m}u^2\frac{\mathrm{d}}{\mathrm{d}\vartheta}\frac{1}{u}\right) = \frac{L^2}{m^2}u^3 - GMu^2$$

which becomes [1]

$$\frac{L^2}{m^2}u^2\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(u^2\frac{-1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\vartheta}\right) = \frac{L^2}{m^2}u^3 - GMu^2$$

hence [1]

$$-\frac{L^2}{m^2}u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\vartheta^2} = \frac{L^2}{m^2}u^3 - GMu^2$$

and thus [1]

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\vartheta^2} = -u + \frac{GMm^2}{L^2}.$$ *[4]*

(f) Hence show that the comet will trace out a path $r(\vartheta)$ of the form

$$r = \frac{L^2}{GMm^2\,(1 + \alpha\cos\vartheta)}$$

where

$$\alpha^2 = 1 + \frac{2L^2\mathcal{E}}{(GMm)^2\,m}.$$ [4]

The expression of part (e) defines simple harmonic motion of u about the point $u = GMm^2/L^2$. This may be shown explicitly by substituting the form given into the equation of motion [2]:

$$\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2}\frac{GMm^2\,(1+\alpha\cos\vartheta)}{L^2} = -\frac{GMm^2\,(\alpha\cos\vartheta)}{L^2} = -\frac{GMm^2\,(1+\alpha\cos\vartheta)}{L^2} + \frac{GMm^2}{L^2}.$$ *[2]*

Adding the maximum and minimum values of r we find the length of the major axis, and equate this to the result from part (c) [1]:

$$2a = \frac{L^2}{GMm^2}\frac{(1+\alpha)+(1-\alpha)}{(1+\alpha)(1-\alpha)} = \frac{L^2}{GMm^2}\frac{2}{1-\alpha^2} = \frac{GMm}{-\mathcal{E}}$$

hence [1]

$$\alpha^2 = 1 + \frac{2L^2\mathcal{E}}{(GMm)^2\,m}.$$ *[2]*

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B4. (a) State the relationship between torque and angular momentum, and explain what is meant by *precession* in the context of rotational motion. Give an example of precession, and state the physical principle from which it results. [5]

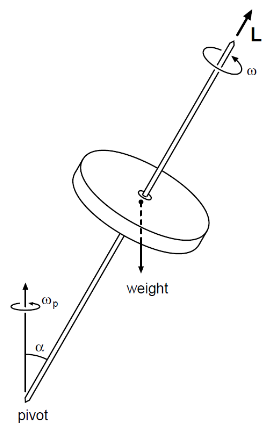
The torque $\boldsymbol{\tau}$ applied to a system is equal to the rate of change of its angular momentum $\mathbf{L}$,

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}. \qquad [\ 1\ ]$$

Precession is the variation of the axis of rotation of a spinning body [1] due to the application of a torque about a different axis [1]. It is apparent in the toppling of a spinning top, the coupling of atomic angular momenta, and the precession of the Earth's axis (precession of the equinoxes) around the ecliptic pole [1]. It results from the conservation of angular momentum [1] in a system in which the precessing object is coupled to other rotational motions. *[4]*

A spinning top comprises a ring of mass M that is connected by spokes of negligible mass to a light axle along the axis of rotational symmetry, about which the top has a moment of inertia I. The axle is suspended at a pivot a distance a from the ring's centre of mass. When the top spins with angular velocity ω the axle assumes a constant angle α to the vertical.

(b) Draw a suitably labelled diagram illustrating this situation. [2]



The diagram should show the ring, axle and pivot, the distance a (omitted above), angle α, and indicate the rotation. *[2]*

(c) Show that the moment of the disc's weight about the support is

$$Mg\,a\sin\alpha$$

and hence that, if α is constant, the spinning disc precesses about the support with angular frequency

$$\Omega = \frac{Mg\,a}{I\omega}.$$ [4]

If $\mathbf{a}$ *is the vector from the pivot to the centre of mass, then the moment of the weight* Mg *about the pivot will be [the vector form is not required for the mark]*

$$|\mathbf{a}\times M\mathbf{g}| = a\sin\alpha Mg.$$ *[1]*

If the spinning disc precesses with angular frequency Ω *(*ω_{p} *in the figure below), then*

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{\Omega}\times\mathbf{L} = \Omega L\sin\alpha$$ *[1]*

where $L = I\omega$ *is the angular momentum of the spinning disc. Since for constant* α *the moment of the weight is the torque applied to the spinning disc, this equals* $\mathrm{d}\mathbf{L}/\mathrm{d}t$*, so*

$$Mg\,a\sin\alpha = \Omega L\sin\alpha = \Omega I\omega\sin\alpha$$ *[1]*

from which

$$\Omega = \frac{Mg\,a}{I\omega}.$$ *[1]*

(d) The ring, of uniform thickness d, has an inner radius p and an outer radius q. Show that, in terms of its total mass M, its moment of inertia is

$$I = \frac{1}{2}M\left(p^2+q^2\right).$$ [4]

We may divide the ring into thin-walled tubes of radius r*, radial thickness* $\mathrm{d}r$ *and density* ρ *per unit area, each of which will have a moment of inertia*

$$I(r)\mathrm{d}r = (\rho 2\pi r\mathrm{d}r)\,r^2.$$ *[1]*

The total moment of inertia is found by integrating this from the inside to the outside of the ring:

$$I = \int_p^q \rho 2\pi r^3\mathrm{d}r = 2\pi\rho\left[\frac{r^4}{4}\right]_p^q = \frac{\pi\rho}{2}\left(q^4-p^4\right) = \frac{\pi\rho}{2}\left(q^2-p^2\right)\left(q^2+p^2\right).$$ *[1]*

The total mass is similarly found to be

$$M = \int_p^q \rho 2\pi r\mathrm{d}r = 2\pi\rho\left[\frac{r^2}{2}\right]_p^q = \pi\rho\left(q^2-p^2\right).$$ *[1]*

TURN OVER

The moment of inertia is hence

$$I = \frac{1}{2} M \left(p^2 + q^2\right).$$ *[1]*

(e) Hence find the time $T = 2\pi/\Omega$ it takes for the top to precess around a vertical axis if $p = 8$ cm, $q = 10$ cm, $d = 1$ cm, $a = 0.1$ m, $M = 0.3$ kg and the top spins at 5 revolutions per second. [2]

From the results of parts (b) and (c), we may write the precession period [1]

$$T \equiv \frac{2\pi}{\Omega} = \frac{2\pi I \omega}{Mga} = \frac{\pi M \left(p^2 + q^2\right) \omega}{Mga} = \frac{\pi \left(p^2 + q^2\right) \omega}{ga}.$$

Substituting the values given, we obtain [1]

$$T = \frac{\pi \left(0.1^2 + 0.08^2\right) \mathrm{m}^2 \, 2\pi \times 5 \, \mathrm{s}^{-1}}{9.8 \, \mathrm{m \, s}^{-2} \, 0.1 \, \mathrm{m}} = 1.65 \, \mathrm{s}.$$ *[2]*

(f) The top is temporarily prevented from precessing by blocking the path of the axle, while the top continues to rotate at an angle α to the vertical with angular frequency ω. The top is then released. Describe its subsequent motion. [3]

The top will not precess immediately with the angular frequency Ω calculated in part (b), for this would imply instantaneous acceleration of its centre of mass [1]. While the rate of precession is being established, the top will begin to fall under gravity, and its centre of mass will acquire a downward velocity component that persists even when the force is cancelled, causing the top's descent to overshoot. The top will thus nod up and down as it precesses [1], in a motion known as nutation*. The initial angular momentum, comprising the top's spin and the precession of its centre of mass, is conserved, but angular momentum is exchanged between these two components [1].* *[3]*

END OF PAPER

