SEMESTER 2 EXAMINATION 2018-2019

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct 'Word to Word' translation dictionary AND it contains no notes, additions or annotations.

Section A

A1. State the parallel and perpendicular axis theorems, explaining clearly the situations to which they apply and any terms or symbols used. [4]

Parallel axis theorem: for any body [0.5], if the moment of inertia I_{CM} is known about an axis through the centre of mass, then the moment of inertia I about a displaced parallel axis [0.5] will be given by *[1]*

$$I = I_{CM} + M\,d^2,$$

where M is the total mass of the body and d the displacement of the axes. *[1]*

Perpendicular axis theorem: for flat (plate-like) objects [0.5], the moment of inertia I about an arbitrary axis perpendicular to the plate [0.5] will be given by the sum of the moments of inertia about orthogonal axes, within (parallel to) the plate, that pass through the same point, e.g. if the plate lies in the $x-y$ plane, *[1]*

$$I = I_x + I_y,$$

where I_x and I_y are the moments of inertia about the axes within the plane. *[1]*

A2. State Kepler's laws and outline the physical assumptions upon which they are based. [4]

Kepler's laws of planetary motion are

(a). The orbit of a planet is an ellipse with the Sun at one of its foci *[1]*

(b). The line from the Sun to the planet sweeps out equal areas in equal intervals of time *[1]*

(c). The square of the orbital period is proportional to the cube of the length of the semi-major axis of its orbit *[1]*

Kepler's laws assume conservation of energy, conservation of angular momentum, non-relativistic motion, and Euclidean space; it may also/alternatively be mentioned that they assume gravity to be a central, conservative force obeying the inverse square law, and that no external forces/torques act. In the stated form, $M \gg m$ and any tidal forces or effects of non-sphericity are neglected. *[1]*

A3. Define torque, show how it may be expressed as a vector product, and state how it affects the angular momentum of a body upon which it acts. [2]

Hence show that if a particle orbiting a point is subject only to a central force directed towards that point, its angular momentum is conserved, irrespective of the shape of the orbit. [2]

Torque $\boldsymbol{\tau}$ is the moment, about a given point, of a force $\mathbf{F}$ applied at a relative position $\mathbf{r}$. Mathematically,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$ *[1]*

Its effect is to change the vector angular momentum $\mathbf{L}$ of the object to which it is applied, according to

$$\boldsymbol{\tau} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}.$$ *[1]*

For a central force, $\mathbf{F}$ is parallel to $\mathbf{r}$ (e.g. $\mathbf{F} = F\hat{\mathbf{r}}$), so the torque will be zero (since $\mathbf{r} \times \hat{\mathbf{r}} = 0$). *[1]*

The angular momentum will hence be constant, so is conserved. *[1]*

A4. Give a formula for the *reduced mass* of a system of two particles of masses m_1 and m_2. For what situations is it a useful quantity? [2]

Calculate the reduced mass for the Earth-Moon system, given that the masses of the Earth and Moon are 5.972×10^{24} kg and 7.346×10^{22} kg respectively. [2]

The reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$ *[1]*

is useful for determining the motions of two particles around each other in the absence of other neighbouring bodies or external forces. *[1]*

For the Earth-Moon system,

$$\mu = \frac{597.2 \times 7.346}{597.2 + 7.346} \times 10^{22}\,\text{kg} = 7.257 \times 10^{22}\,\text{kg}.$$ *[2]*

A5. Explain the physical principles behind *Buys Ballot's law*: that, if you stand with your back to the wind in the northern hemisphere, the atmospheric pressure will be lower on your left than to your right. [4]

The rotation of the Earth results in velocity-dependent Coriolis forces upon the atmosphere as it responds to differences in pressure [1]. These cause any horizontally-moving parcel of air, away from the pole and equator in the northern hemisphere, to experience a force to its right [1]. For motion along a straight-ish line, the Coriolis force has to be balanced by a pressure gradient from right to left, so that

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there is a greater hydrostatic force upon the right of the parcel than the left [1]. If you have your back to the wind, you are facing in the direction of motion of the parcel, so the pressure will be lower to your left than to your right [1]. *[4]*

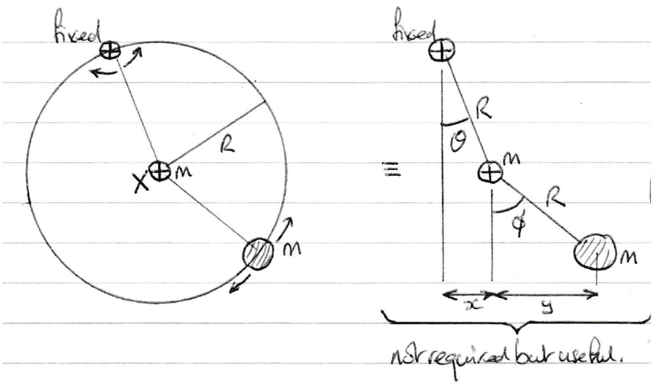
Section B

B1. (a) Explain what is meant by a *normal mode* of an oscillating system. [2]

A normal mode is a motion in which all parts of the system oscillate with the same single frequency and (therefore) with a fixed phase relationship between each other; the amplitudes may differ, but maintain the same proportions. *[2]*

A bead of mass m slides freely on a smooth circular ring of negligible mass and radius R, which is attached at its centre to a small hub of mass m. The ring is free to rotate in its own plane about a fixed axis through a point on its circumference, and the system makes *small* oscillations under gravity.

(b) Sketch the system described. [2]



[2]

(c) For the ring position shown in your diagram, indicate the point about which the bead rotates. [1]

The bead rotates about the centre of the ring, labelled X in the figure above. *[1]*

(d) Derive the matrix equation governing the angular coordinates ϑ and φ of the ring relative to the pivot, and bead relative to the ring, respectively,

$$\frac{d^2}{dt^2}\begin{pmatrix}\vartheta\\ \varphi\end{pmatrix} = \frac{g}{R}\begin{pmatrix}-2 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}\vartheta\\ \varphi\end{pmatrix}.$$ [6]

The bead is supported by a reaction force which, for small angles, will be approximately mg, directed perpendicular to the ring circumference and hence towards the ring centre X. This force *has a horizontal component $mg\sin\varphi \approx mg\varphi$. Since $x \approx R\vartheta$ and $y \approx R\varphi$, Newton's second law gives*

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x+y) = mR\frac{\mathrm{d}^2}{\mathrm{d}^2}(\vartheta+\varphi) = -mg\varphi$$

hence

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\vartheta+\varphi) = -\frac{g}{R}\varphi.$$ *[2]*

Since the ring supports the bead, it is itself supported by a reaction force of approximately $2mg$, and is hence similarly subject to a perpendicular force $2mg\sin\vartheta \approx 2mg\vartheta$, in addition to the weight of the bead. Since $x \approx R\vartheta$, we may write

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = mR\frac{\mathrm{d}^2\vartheta}{\mathrm{d}^2} = -2mg\vartheta + mg\varphi$$

hence

$$\frac{\mathrm{d}^2\vartheta}{\mathrm{d}t^2} = \frac{g}{R}(\varphi - 2\vartheta).$$ *[2]*

Subtracting the second result from the first, we obtain

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = \frac{g}{R}(2\vartheta - 2\varphi).$$ *[1]*

Our two results may now be combined in matrix form,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix}\vartheta\\ \varphi\end{pmatrix} = \frac{g}{R}\begin{pmatrix}-2 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}\vartheta\\ \varphi\end{pmatrix}.$$ *[1]*

(Note that, since $x \approx R\vartheta$ and $y \approx R\varphi$, the same result is obtained if one works in terms of the linear horizontal coordinates x and y.)

(e) Hence derive the eigenvalue matrix equation and solve it to determine the angular frequencies of the two normal modes of the system. [5]

We assume that normal modes take the form

$$\begin{pmatrix}\vartheta\\ \varphi\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix}\exp(\mathrm{i}\omega t).$$ *[1]*

Substituting this into the matrix equation of (d), we find

$$\begin{pmatrix}a\\ b\end{pmatrix}(\mathrm{i}\omega)^2\exp(\mathrm{i}\omega t) = \frac{g}{R}\begin{pmatrix}-2 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\exp(\mathrm{i}\omega t).$$ *[1]*

Cancelling terms and rearranging, we obtain

$$\begin{pmatrix}\omega^2 - \frac{2g}{R} & \frac{g}{R}\\ \frac{2g}{R} & \omega^2 - \frac{2g}{R}\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = 0,$$ *[1]*

which requires that the matrix determinant is zero, i.e.,

$$\left(\omega^2 - \frac{2g}{R}\right)\left(\omega^2 - \frac{2g}{R}\right) - \frac{g}{R}\frac{2g}{R} = 0$$

which yields the quadratic equation

$$\left(\omega^2\right)^2 - \frac{4g}{R}\omega^2 + 2\left(\frac{g}{R}\right)^2 = 0$$ *[1]*

to which the solutions are

$$\omega^2 = \frac{\frac{4g}{R} \pm \sqrt{\left(\frac{4g}{R}\right)^2 - 4 \times 2\left(\frac{g}{R}\right)^2}}{2} = \left(2 \pm \sqrt{2}\right)\frac{g}{R}.$$

The eigenfrequencies are hence

$$\omega = \sqrt{\left(2 \pm \sqrt{2}\right)\frac{g}{R}}.$$ *[1]*

(f) Calculate the (unnormalized) eigenvectors associated with each eigenvalue. [2]

If $\omega^2 = (2 + \sqrt{2})(g/R)$*, our eigenvector equation will be*

$$\begin{pmatrix} (2+\sqrt{2}-2)\frac{g}{R} & \frac{g}{R} \\ 2\frac{g}{R} & (2+\sqrt{2}-2)\frac{g}{R} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

i.e., equivalent equations $\sqrt{2}a + b = 0$ *and* $2a + \sqrt{(2)}b = 0.$ *It follows that* $b = -\sqrt{2}a$ *and that the unnormalized eigenvector will be*

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}.$$ *[1]*

If $\omega^2 = (2 - \sqrt{2})(g/R)$*, we correspondingly find*

$$\begin{pmatrix} (2-\sqrt{2}-2)\frac{g}{R} & \frac{g}{R} \\ 2\frac{g}{R} & (2-\sqrt{2}-2)\frac{g}{R} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

i.e., equivalent equations $-\sqrt{2}a + b = 0$ *and* $2a - \sqrt{(2)}b = 0.$ *It follows that* $b = \sqrt{2}a$ *and that the unnormalized eigenvector will be*

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}.$$ *[1]*

(g) Describe the motions of the ring and bead in each mode. [2]

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At the lower frequency, the bead and ring move in the same direction, in a symmetric motion except that the bead moves with a greater amplitude. *[1]*

At the higher frequency, the bead and ring move in opposite directions, in an antisymmetric motion except that the bead again moves with a greater amplitude. *[1]*

B2. (a) Explain what is meant by an object's *moment of inertia* about an axis, and define it mathematically in terms of the distribution of the object's mass. [2]

The moment of inertia I is the constant of proportionality between the object's angular momentum L and its angular velocity $\dot{\vartheta}$ about that axis,

$$L = I\dot{\vartheta}$$

and hence represents the reluctance of the object to change its rate of rotation in response to an applied torque τ (that is, $\tau = I\ddot{\vartheta}$). *[1]*

The moment of inertia I is defined as

$$I = \int_{\text{object}} r_\perp^2 \, \mathrm{d}m = \int_{\text{object}} \rho(\mathbf{r})\, r_\perp^2 \, \mathrm{d}V$$

where $\mathrm{d}m$ is an element of mass lying a distance $r_\perp$ from the axis of rotation, $\rho(\mathbf{r})$ is the position-dependent density and $\mathrm{d}V$ an element of volume. [Either expression will suffice.] *[1]*

(b) Show that the moment of inertia of a uniform solid cylinder of radius R, length L and mass M is given by $I = \frac{1}{2}MR^2$. [4]

We divide the cylinder most conveniently into concentric cylindrical shells, taken to be of density ρ, and thus obtain

$$I = \int_0^R r^2 \rho(2\pi r L \mathrm{d}r) = 2\pi L\rho \frac{R^4}{4}$$

where the bracketed term is the volume of a cylindrical shell of radius r, length L and thickness $\mathrm{d}r$. *[2]*

Since the volume of the cylinder will be $\pi R^2 L$, the total mass will be

$$M = \pi R^2 L \rho.$$ *[1]*

The moment of inertia is hence

$$I = \frac{1}{2}MR^2.$$ *[1]*

The *'bouncing bomb'* used in the *'Dam Buster'* raids of *Operation Chastise* was a cylinder measuring 1.27 m in diameter by 1.52 m in length, with a mass of 4200 kg. It was carried beneath the bomber aircraft with the axis of the cylinder parallel to the wings, perpendicular to the direction of travel. To stabilize the

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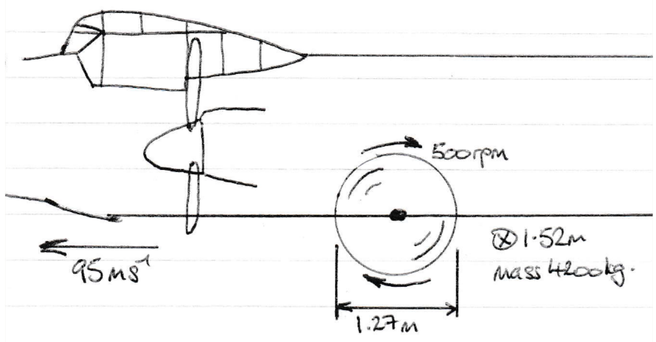
bomb once deployed, it was rotated whilst aboard the aircraft at 500 revolutions per minute, the sense of rotation being clockwise when viewed from the left of the aircraft. The aircraft approached its target at a speed of 95 m s^{-1}.

(c) Sketch the situation described. [2]



[2]

(d) Calculate the speed relative to the ground of the lowest point of the bomb. [1]

The speed of the lowest point will be the sum of that of the aircraft and the rotational speed $r\omega$, where r is the bomb radius and ω its angular velocity. With the values given, this yields a speed

$$95\,\mathrm{m\,s^{-1}} + \frac{1.27}{2}\,\mathrm{m}\frac{2\pi\times 500}{60}\,\mathrm{rad\,s^{-1}} = 128\,\mathrm{m\,s^{-1}}.$$ *[1]*

(e) Calculate the magnitude and direction of the bomb's angular momentum. [3]

Using the result from part (b),

$$I = \frac{1}{2}MR^2 = \frac{1}{2}4200\,\mathrm{kg}\left(\frac{1.27\,\mathrm{m}}{2}\right)^2 = 850\,\mathrm{kg\,m^2}.$$ *[1]*

The angular momentum L is therefore

$$L = I\omega = \frac{1}{2}4200\,\mathrm{kg}\left(\frac{1.27\,\mathrm{m}}{2}\right)^2\frac{2\pi\times 500}{60}\,\mathrm{rad\,s^{-1}} = 44,300\,\mathrm{kg\,m^2\,s^{-1}}.$$ *[1]*

By the right-hand rule, or by explicitly finding $\mathbf{r}\times\mathbf{p}$ for points in the rotating bomb, we find that the angular momentum points along the bomb axis to the right – i.e., into the page in the diagram above. *[1]*

To line up for the approach to the target, the bomber aircraft needed to turn to the right. The pilot therefore lowered the right wing, so that the aircraft initially rotated about a longitudinal (fore-aft) axis at a rate of 10 deg s^{-1}.

(f) Calculate the magnitude and direction of the gyroscopic torque exerted upon the aircraft by the rotating bomb. [4]

The aircraft rolls with an angular velocity $\Omega = (2\pi/360) \times 10$ rad s^{-1}, where lowering the right wing corresponds to the $\mathbf{\Omega}$ vector pointing forwards. The torque required to rotate the bomb's rotational axis and hence angular momentum $\mathbf{L}$ in this way is given by

$$\boldsymbol{\tau} = \frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{\Omega} \times \mathbf{L}.$$ *[1]*

With the values given, we hence find

$$\boldsymbol{\tau} = \frac{2\pi}{360} \times 10\,\mathrm{rad\,s^{-1}} \times 44,300\,\mathrm{kg\,m^2\,s^{-1}}\,\hat{\boldsymbol{\tau}} = 7700\,\mathrm{N\,m}\,\hat{\boldsymbol{\tau}},$$

where the direction of the torque vector $\hat{\boldsymbol{\tau}}$ is given by (forwards) $\times$ (right) – i.e., downwards. *[2]*

To balance this, the torque upon the aircraft will be 7700 N m upwards. *[1]*

(g) What effect will this torque have had upon the aircraft? [2]

The torque will cause the aircraft to yaw, about a vertical axis, to the left – i.e., the opposite direction from the intended turn. *[2]*

(h) How large a force would need to be applied to the tail of the aircraft to counteract this effect? You may assume the tail to be about 11 m from the centre of mass of the aircraft. [1]

The torque τ could be counteracted by the application of a force F at a distance d where $\tau = Fd$, so the required force will be

$$F = \frac{\tau}{d} = \frac{7700\,\mathrm{N\,m}}{11\,\mathrm{m}} = 700\,\mathrm{N}.$$ *[1]*

(i) In which direction should this force be applied? [1]

To produce a downward torque vector, the force should push the tail of the aircraft to the left. *[1]*

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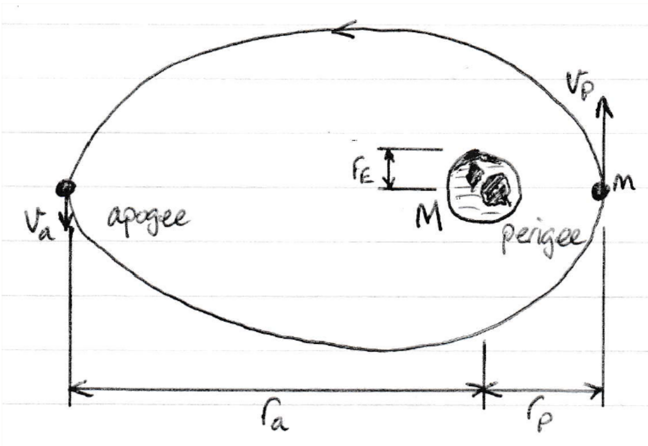
B3. (a) State the relationship between the acceleration g due to gravity at the Earth's surface, Newton's gravitational constant, and the mass and radius of the Earth (assuming it to be spherically symmetric). Define any symbols used. [2]

$$g = \frac{GM}{R^2}$$ *[1]*

where G is Newton's gravitational constant, M is the Earth's mass and R the Earth's radius. *[1]*

A Galileo GNSS satellite is launched by rocket into orbit around the Earth. After the first boost stage, the rocket is 1130 km above the Earth's surface and has a velocity of 9.2 km s^{-1} perpendicular to a line from the Earth's centre.

(b) Sketch the situation described, and the orbit established after the first boost stage. [3]



[3]

(c) Show, by considering two conservation laws, that the furthest distance of the satellite from the Earth's centre during the subsequent orbital motion may be written as

$$r_a = \frac{r_p}{\frac{2GM}{r_p v_p^2} - 1},$$

where G is the gravitational constant, M the mass of the Earth, and r_p and v_p are respectively the radial distance from the centre of the Earth and the rocket's velocity immediately after the first boost stage. [8]

For conservation of angular momentum,

$$r_p v_p = r_a v_a \qquad [\,1\,]$$

and for conservation of energy

$$m\left(\frac{1}{2}v_p^2 - \frac{GM}{r_p}\right) = m\left(\frac{1}{2}v_a^2 - \frac{GM}{r_a}\right), \qquad [\,1\,]$$

where M and m are the masses of the Earth and rocket respectively, and r_p, v_p, r_a and v_p are the distances and velocities at the perigee and apogee as shown in the diagram.

To find r_a, we combine these expressions to eliminate v_a:

$$v_a = \frac{r_p v_p}{r_a} \qquad [\,1\,]$$

so

$$\frac{1}{2}v_p^2 - \frac{GM}{r_p} = \frac{1}{2}\left(\frac{r_p v_p}{r_a}\right)^2 - \frac{GM}{r_a}. \qquad [\,1\,]$$

Collecting terms in v_p and GM,

$$\frac{1}{2}v_p^2\left[1 - \left(\frac{r_p}{r_a}\right)^2\right] = -GM\left(\frac{1}{r_a} - \frac{1}{r_p}\right) \qquad [\,1\,]$$

hence

$$v_p^2\left(r_a^2 - r_p^2\right) = -2GM\frac{r_a}{r_p}\left(r_p - r_a\right) \qquad [\,1\,]$$

so

$$v_p^2\left(r_a + r_p\right) = 2GM\frac{r_a}{r_p}. \qquad [\,1\,]$$

Collecting terms in r_a and rearranging now yields

$$r_a = \frac{r_p v_p^2}{\frac{2GM}{r_p} - v_p^2} = \frac{r_p}{\frac{2GM}{r_p v_p^2} - 1}. \qquad [\,1\,]$$

(d) By expressing GM in terms of the Earth's radius r_E and the gravitational acceleration g at its surface, show that the furthest distance of the satellite may be written as

$$r_a = \frac{r_p}{\frac{2g\,r_E^2}{r_p v_p^2} - 1}. \qquad [\,1\,]$$

We use the result of part (a) to write $g = GM/r_E^2$, where r_E is the Earth's radius, and hence

$$GM = g\,r_E^2$$

giving

$$r_a = \frac{r_p v_p^2}{\frac{2GM}{r_p} - v_p^2} = \frac{r_p}{\frac{2g\,r_E^2}{r_p v_p^2} - 1}. \qquad [\,1\,]$$

(e) Calculate the numerical value of this distance. [1]

With the values given, this yields

$$r_a = \frac{(6370 + 1130)\,\mathrm{km}}{\frac{2\times 9.81\,\mathrm{m\,s^{-2}}\times(6370\times 10^3\,\mathrm{m})^2}{(6370+1130)\times 10^3\,\mathrm{m}\times(9200\,\mathrm{m\,s^{-1}})^2} - 1} = 29\,500\,\mathrm{km}. \quad [\,1\,]$$

(f) Derive the value of the eccentricity of the orbit. [5]

In terms of the length a of the semi-major axis, and the eccentricity e,

$$r_p = (1-e)a;$$

and

$$r_a = (1+e)a. \quad [\,1\,]$$

We eliminate a by writing e.g.,

$$a = \frac{r_p}{1-e} \quad [\,1\,]$$

and substituting to give

$$r_a = \frac{1+e}{1-e} r_p$$

hence

$$(1-e)r_a = (1+e)r_p \quad [\,1\,]$$

so, collecting terms in e,

$$e(r_a + r_p) = r_a - r_p$$

and thus

$$e = \frac{r_a - r_p}{r_a + r_p}. \quad [\,1\,]$$

With the values given,

$$e = \frac{29500 - (6370 + 1130)}{29500 + (6370 + 1130)} = 0.595. \quad [\,1\,]$$

The Earth's radius may be taken to be 6370 km, and gravitational acceleration at the Earth's surface to be 9.81 m s^{-1}.

B4. (a) Show that, if a fixed-length vector $\mathbf{A}$ rotates with angular velocity ω about an axis defined by the vector $\hat{\omega}$, and we define $\boldsymbol{\omega} \equiv \omega\hat{\omega}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}. \qquad [\,4\,]$$

From a suitable diagram, considering the magnitude and direction of the change of vector, we see that the infinitessimal change $\mathrm{d}\mathbf{A}$ resulting from rotation of $\mathbf{A}$ through an infinitessimal angle $\mathrm{d}\varphi$ about $\hat{\omega}$ will be

$$\mathrm{d}\mathbf{A} = \hat{\omega} \times \mathbf{A}\mathrm{d}\varphi. \qquad [\,2\,]$$

Dividing by an infintessimal timestep $\mathrm{d}t$ and noting that the angular velocity $\omega \equiv \mathrm{d}\varphi/\mathrm{d}t$,

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \hat{\omega} \times \mathbf{A}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega\hat{\omega} \times \mathbf{A} = \boldsymbol{\omega} \times \mathbf{A}. \qquad [\,2\,]$$

The unit vectors $\hat{\mathbf{i}}'$, $\hat{\mathbf{j}}'$ and $\hat{\mathbf{k}}'$ of a rotating coordinate frame rotate with angular velocity ω about an axis $\hat{\omega}$, so that a vector $\mathbf{a} \equiv a_i\hat{\mathbf{i}} + a_j\hat{\mathbf{j}} + a_k\hat{\mathbf{k}}$ in an inertial frame $\left\{\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}\right\}$ may be written at a given time as $\mathbf{b} \equiv b_i\hat{\mathbf{i}}' + b_j\hat{\mathbf{j}}' + b_k\hat{\mathbf{k}}'$.

(b) Show that

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} = \dot{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}$$

and hence that

$$\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}t^2} = \ddot{\mathbf{b}} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}),$$

where $\dot{\mathbf{b}} \equiv \dot{b}_i\hat{\mathbf{i}}' + \dot{b}_j\hat{\mathbf{j}}' + \dot{b}_k\hat{\mathbf{k}}'$, $\ddot{\mathbf{b}} \equiv \ddot{b}_i\hat{\mathbf{i}}' + \ddot{b}_j\hat{\mathbf{j}}' + \ddot{b}_k\hat{\mathbf{k}}'$, and $\dot{b}_i \equiv \mathrm{d}b_i/\mathrm{d}t$ etc. [6]

In an inertial frame, the unit vectors of the rotating frame change with time, so the vector must be differentiated as a product [1 mark per line]: [3]

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} &= \left(\frac{\mathrm{d}b_i}{\mathrm{d}t}\hat{\mathbf{i}}' + b_i\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t}\right) + \left(\frac{\mathrm{d}b_j}{\mathrm{d}t}\hat{\mathbf{j}}' + b_j\frac{\mathrm{d}\hat{\mathbf{j}}'}{\mathrm{d}t}\right) + \left(\frac{\mathrm{d}b_k}{\mathrm{d}t}\hat{\mathbf{k}}' + b_k\frac{\mathrm{d}\hat{\mathbf{k}}'}{\mathrm{d}t}\right) \\
&= \dot{b}_i\hat{\mathbf{i}}' + \dot{b}_j\hat{\mathbf{j}}' + \dot{b}_k\hat{\mathbf{k}}' + b_i\boldsymbol{\omega} \times \hat{\mathbf{i}}' + b_j\boldsymbol{\omega} \times \hat{\mathbf{j}}' + b_k\boldsymbol{\omega} \times \hat{\mathbf{k}}' \\
&= \dot{b}_i\hat{\mathbf{i}}' + \dot{b}_j\hat{\mathbf{j}}' + \dot{b}_k\hat{\mathbf{k}}' + \boldsymbol{\omega} \times \mathbf{b} \equiv \dot{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}.
\end{aligned}$$

Differentiating a second time, noting that the vectors $\mathbf{a}$ and $\mathbf{b}$ are equivalent [1 mark per line], [3]

$$\begin{aligned}
\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}t^2} &= \left(\ddot{b}_i\hat{\mathbf{i}}' + \dot{b}_i\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t}\right) + \left(\ddot{b}_j\hat{\mathbf{j}}' + \dot{b}_i\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t}\right) + \left(\ddot{b}_k\hat{\mathbf{k}}' + \dot{b}_i\frac{\mathrm{d}\hat{\mathbf{i}}'}{\mathrm{d}t}\right) + \boldsymbol{\omega} \times \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}t} \\
&= \ddot{b}_i\hat{\mathbf{i}}' + \ddot{b}_j\hat{\mathbf{j}}' + \ddot{b}_k\hat{\mathbf{k}}' + \dot{b}_i\boldsymbol{\omega} \times \hat{\mathbf{i}}' + \dot{b}_j\boldsymbol{\omega} \times \hat{\mathbf{j}}' + \dot{b}_k\boldsymbol{\omega} \times \hat{\mathbf{k}}' + \boldsymbol{\omega} \times \left(\dot{\mathbf{b}} + \boldsymbol{\omega} \times \mathbf{b}\right) \\
&= \ddot{\mathbf{b}} + \boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) = \ddot{\mathbf{b}} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}).
\end{aligned}$$

TURN OVER

(c) Hence show that, for a particle of mass m subject to gravitational acceleration $\mathbf{g}$ and an applied force $\mathbf{F}$, the equation of motion in the rotating frame will be

$$m\ddot{\mathbf{b}} = \mathbf{F} + m\mathbf{g} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) - 2m\boldsymbol{\omega} \times \dot{\mathbf{b}}.$$ [2]

According to Newton's second law, the total force $\mathbf{F} + m\mathbf{g} = m\mathrm{d}^2\mathbf{a}/\mathrm{d}t^2$. Substituting the result above and rearranging, *[2]*

$$\begin{aligned} \mathbf{F} + m\mathbf{g} &= m\frac{\mathrm{d}^2\mathbf{a}}{\mathrm{d}t^2} = m\ddot{\mathbf{b}} + 2m\boldsymbol{\omega} \times \dot{\mathbf{b}} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \\ \Rightarrow \qquad m\ddot{\mathbf{b}} &= \mathbf{F} + m\mathbf{g} - 2m\boldsymbol{\omega} \times \dot{\mathbf{b}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \,. \end{aligned}$$

The *vibrating structure gyroscope* comprises a miniature tuning fork, which may be taken to lie in the $x'-z'$ plane with the z'-axis following the axis of symmetry. The prongs of the tuning fork are driven in opposite directions in the x' direction so that their displacements at time t are $x' = \pm x'_0 \sin(\omega_0 t)$.

(d) Show that, if the gyroscope rotates about an angular velocity vector $\boldsymbol{\Omega}$ with components $\Omega_{x'}, \Omega_{y'}, \Omega_{z'}$, then the prongs will experience Coriolis forces

$$\mathbf{F}_{Cor} = \mp 2mx'_0\omega_0 \cos(\omega_0 t)\, \boldsymbol{\Omega} \times \hat{\mathbf{i}}',$$

where m is the effective mass of each prong. [3]

The Coriolis force is the final, velocity-dependent term in the expression of part (c). Writing the prong positions as

$$\mathbf{b} = \pm\mathbf{b_0} + x'\hat{\mathbf{i}}'$$ *[1]*

where the vector joining the rest positions of the prongs is $\mathbf{b_0}$, we find

$$\dot{\mathbf{b}} = \pm x'_0\omega_0 \cos(\omega_0 t)\hat{\mathbf{i}}'$$ *[1]*

and hence the Coriolis forces

$$-2m\boldsymbol{\Omega} \times \dot{\mathbf{b}} = \mp 2mx'_0\omega_0 \cos(\omega_0 t)\, \boldsymbol{\Omega} \times \hat{\mathbf{i}}'.$$ *[1]*

(e) Hence show that in the y' direction, the relative displacement of the prongs, neglecting any resonance effects, will be

$$\Delta y' = \frac{4x_0'}{\omega_0}\Omega_{z'}\cos(\omega_0 t). \qquad [\,3\,]$$

From our answer to (d), the Coriolis acceleration will be

$$-2\boldsymbol{\Omega}\times\dot{\mathbf{b}} = \mp 2x_0'\omega_0\cos(\omega_0 t)\,\boldsymbol{\Omega}\times\hat{\mathbf{i}}'. \qquad [\,1\,]$$

Integrating this twice and evaluating the vector product, we obtain the displacement due to the Coriolis force,

$$\mp 2x_0'\omega_0 \iint \cos(\omega_0 t)\,\mathrm{d}t\,\mathrm{d}t\;\boldsymbol{\Omega}\times\hat{\mathbf{i}}' = \pm\frac{2x_0'}{\omega_0}\cos(\omega_0 t)\;\boldsymbol{\Omega}\times\hat{\mathbf{i}}' = \pm\frac{2x_0'}{\omega_0}\cos(\omega_0 t)\;\Omega_{z'}\hat{\mathbf{j}}'. \qquad [\,1\,]$$

The relative displacement in the y' direction of the two prongs will be the difference between these two displacements, i.e.,

$$\Delta y' = \frac{4x_0'}{\omega_0}\Omega_{z'}\cos(\omega_0 t). \qquad [\,1\,]$$

(f) If the device is driven at a frequency of 15 kHz with amplitude 5 μm, find the amplitude of the relative motion when the device is used to measure the rotation of a Formula 1 engine at 10,000 rpm (revolutions per minute). [2]

With the values given, the amplitude of the relative motion will be

$$\frac{4x_0'}{\omega_0}\Omega_{z'} = \frac{4\times 5\times 10^{-6}\,\mathrm{m}}{2\pi\times 15\,000\,\mathrm{rad\,s^{-1}}}\,\frac{2\pi\times 10\,000}{60\,\mathrm{s}} = 220\,\mathrm{nm}. \qquad [\,2\,]$$

END OF PAPER