SEMESTER 1 EXAMINATION 2019-2020

CLASSICAL MECHANICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language dictionary is permitted ONLY IF it is a paper version of a direct 'Word to Word' translation dictionary AND it contains no notes, additions or annotations.

Section A

A1. What is meant by a *central* force? Give two examples of such forces. [2]

Why can a central force between two objects not affect the angular momentum of one about the other? [2]

A central force is one that acts along the straight line joining the two interacting objects. *[1]*

Examples include gravity, Coulomb electrostatic forces, the tension transmitted by a taut rod such as the spoke of a bicycle wheel [0.5 each]. *[1]*

To change the angular momentum requires a torque [1]. If the force $\mathbf{F}$ is central, it will act parallel to the position vector $\mathbf{r}$, so the torque $\boldsymbol{\tau} = \mathbf{F} \times \mathbf{r}$ will be zero [1]. *[2]*

A2. State Kepler's laws and outline the physical assumptions that underly them. [4]

Kepler's laws of planetary motion are

(a). The orbit of a planet is an ellipse with the Sun at one of its foci *[1]*

(b). The line from the Sun to the planet sweeps out equal areas in equal intervals of time *[1]*

(c). The square of the orbital period is proportional to the cube of the semi-major axis of its orbit *[1]*

Kepler's laws assume conservation of energy, conservation of angular momentum, and non-relativistic, Euclidean space; it may also/alternatively be mentioned that they assume gravity to be a central force obeying the inverse square law, and that no external forces/torques act. In the stated form, $M \gg m$ and any tidal forces or effects of non-sphericity are neglected. *[1]*

A3. Show that, if a spacecraft of total mass $m(t)$ propels itself by ejecting exhaust gas from its rocket motor with a relative velocity $\mathbf{u}$, then its velocity $\mathbf{v}(t)$ satisfies

$$m\mathrm{d}\mathbf{v} = -\mathbf{u}\mathrm{d}m,$$ [2]

and hence, making clear any assumptions in your derivation, that the initial and final speeds v_i and v_f are related to the initial and final masses m_i and m_f by

$$v_f = v_i + u \ln \frac{m_i}{m_f}.$$ [2]

Equating the total momenta of the spacecraft and exhaust before and after ejection of an infinitessimal mass $\mathrm{d}m$ *that results in a velocity increase* $\mathrm{d}\mathbf{v}$*,*

$$m\mathbf{v} = (m - \mathrm{d}m)(\mathbf{v} + \mathrm{d}\mathbf{v}) + \mathrm{d}m(\mathbf{v} + \mathbf{u}).$$ *[1]*

Expanding this expression, cancelling terms, and neglecting the term $\mathrm{d}m\mathrm{d}\mathbf{v}$*, which will be of vanishing significance for infinitessimal changes, we obtain*

$$m\mathrm{d}\mathbf{v} = -\mathbf{u}\mathrm{d}m.$$ *[1]*

Assuming $\mathbf{u}$ *and* $\mathbf{v}$ *to be aligned, this expression may be rearranged to give*

$$\mathrm{d}v = -u\frac{\mathrm{d}m}{m}$$ *[1]*

which can be integrated to give

$$v_f - v_i = -u\left(\ln m_f - \ln m_i\right)$$

hence

$$v_f = v_i + u\ln\frac{m_i}{m_f}.$$ *[1]*

A4. Instead of using jet thrusters to rotate a spacecraft, an engineer proposes using the reaction obtained when using an electric motor, attached to the spacecraft, to rotate a flywheel. Explain, with reference to physical laws, why this will work. What must be done to a flywheel with moment of inertia I_f in order to rotate the spacecraft of moment of inertia I_s through 90 degrees? [4]

Conservation of angular momentum for the isolated spacecraft+flywheel combination means that if the flywheel is rotated in one direction, the spacecraft must rotate in the opposite direction about the same axis. *[1]*

The angular velocities of the flywheel and spacecraft are at any time related by

$$I_f\omega_f(t) + I_s\omega_s(t) = 0.$$ *[1]*

Assuming for simplicity that all rotation is about a constant axis, the angle through which the spacecraft rotates may be written as

$$\vartheta_s = \int \omega_s(t)\mathrm{d}t = \frac{I_f}{I_s}\int \omega_f(t)\mathrm{d}t = \frac{I_f}{I_s}\vartheta_f,$$ *[1]*

so, to rotate the spacecraft through 90°, the flywheel must be rotated through an angle $(I_s/I_f) \times 90°$*.* *[1]*

TURN OVER

A5. At a rifle range in Tasmania (41° S), a rifle bullet is fired with an initial speed (*muzzle velocity*) v horizontally towards the west. Explain how, and in which direction, it is deflected as a result of the Earth's rotation. [4]

In the frame of the shooting range, the bullet experiences gravity, centrifugal and Coriolis forces, which reflect that the freely-moving bullet will continue in a straight line as the Earth rotates beneath it. *[1]*

Relative to the bullet, the shooting range will move along a circle perpendicular to the polar axis. Since the Earth rotates from west to east, the shooting range will rise towards a westward-moving bullet, which will hence appear to move downwards as a result of the Earth's rotation. *[1]*

The downward direction perpendicular to the polar axis will, for a point in the southern hemisphere, be angled towards the south by an angle equal to the latitude of the point. *[1]*

The bullet will hence be deflected downwards and to the left (south), at an angle of 41° to the vertical. *[1]*

Section B

B1. (a) Explain how a rotation angle and axis can be represented by a vector $\boldsymbol{\varphi}$. [2]

A unit vector $\hat{\boldsymbol{\varphi}}$ is defined to be parallel to the rotation axis and orientated according to the right-hand rule [1]. The rotation vector $\boldsymbol{\varphi} \equiv \varphi\hat{\boldsymbol{\varphi}}$ is then formed by multiplying the unit vector $\hat{\boldsymbol{\varphi}}$ that defines the axis by the angle φ of the rotation [1]. *[2]*

(b) Demonstrate, with an example, that rotations through finite angles do not in general commute – i.e., $\boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 \neq \boldsymbol{\varphi}_2 + \boldsymbol{\varphi}_1$. [1]

If a postcard initially lies on a table with its picture side up and correctly orientated, rotation through 180° about a horizontal transverse axis (a) will show the address and message inverted and clockwise rotation though 90° about a vertical axis will leave the top of the postcard on the left. If the rotations are carried out in the opposite order (b), the top of the postcard will end on the right. *[1]*

(c) Show that, if a fixed-length vector $\mathbf{A}$ rotates with angular velocity ω about an axis defined by the vector $\hat{\boldsymbol{\omega}}$, and we define $\boldsymbol{\omega} \equiv \omega\hat{\boldsymbol{\omega}}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{A}.$$ [4]

The infinitessimal change $\mathrm{d}\mathbf{A}$ resulting from rotation of $\mathbf{A}$ through an infinitessimal angle $\mathrm{d}\varphi$ about $\hat{\boldsymbol{\omega}}$ will be of magnitude $A \sin\vartheta\, \mathrm{d}\varphi$, where ϑ is the angle between $\mathbf{A}$ and $\hat{\boldsymbol{\omega}}$ [1] and, as its direction is perpendicular to both $\mathbf{A}$ and $\hat{\boldsymbol{\omega}}$, it may be written, assuming right-hand rotation about $\hat{\boldsymbol{\omega}}$, as [1]

$$\mathrm{d}\mathbf{A} = \hat{\boldsymbol{\omega}} \times \mathbf{A}\mathrm{d}\varphi.$$ *[2]*

Dividing by an infintessimal timestep $\mathrm{d}t$ [1] and noting that the angular velocity $\omega \equiv \mathrm{d}\varphi/\mathrm{d}t$ [1],

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \hat{\boldsymbol{\omega}} \times \mathbf{A}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega\hat{\boldsymbol{\omega}} \times \mathbf{A} = \boldsymbol{\omega} \times \mathbf{A}.$$ *[2]*

TURN OVER

A thin coin of radius a and mass m is set spinning upon a smooth table. After spinning on its edge for a while, the coin begins to topple, and establishes a motion in which its rotational symmetry axis $\hat{\boldsymbol{n}}$ precesses about the vertical $\hat{\boldsymbol{\Omega}}$ at a constant angle ϑ. The rolling motion is a combination of rotation at rate Ω about $\hat{\boldsymbol{\Omega}}$ and rotation at rate ω_n about $\hat{\boldsymbol{n}}$. As the coin does not slip where it touches the table, this corresponds instantaneously to rotation about a diameter $\hat{\boldsymbol{d}}$ through this point, where $\Omega\hat{\boldsymbol{\Omega}} - \omega_n\hat{\boldsymbol{n}} = \omega_d\hat{\mathbf{d}}$ and $\omega_n = \Omega\cos\vartheta$.

(d) Show that the coin's moment of inertia about $\hat{\mathbf{d}}$ is $ma^2/4$. [4]

The most elegant solution is to calculate the moment of inertia about the rotational symmetry axis and use the perpendicular axis theorem to deduce the moment of inertia about a diameter. The coin has a mass density $\rho = m/(\pi a^2)$ per unit area [1], so a ring of radius r and thickness $\mathrm{d}r$ will have mass $2\pi r\rho\mathrm{d}r$ [1], and the total moment of inertia about the symmetry axis will be *[2]*

$$I_s = \int_0^a r^2 2\pi r\rho\mathrm{d}r = \left[\frac{2\pi\rho r^4}{4}\right]_0^a = \frac{2\pi\rho a^4}{4} = \frac{ma^2}{2}.$$ *[1]*

By the perpendicular axis theorem, $I_s = I_d + I_d = 2I_d$, hence

$$I_d = \frac{ma^2}{4}.$$ *[1]*

(e) Show that the coin's angular momentum $\mathbf{L}$ satisfies

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \Omega\,\hat{\boldsymbol{\Omega}} \times \mathbf{L} = mga\cos\vartheta\,\hat{\mathbf{h}}$$

where Ω is the angular precession frequency, $\hat{\boldsymbol{\Omega}}$ is a vertical unit vector, and $\hat{\mathbf{h}}$ is a unit vector parallel to the horizontal diameter of the coin. [4]

Application of part (c) to the angular momentum [1], and separating the precession frequency vector into its magnitude and unit vector [1], gives

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \mathbf{L} = \Omega\hat{\boldsymbol{\Omega}} \times \mathbf{L}.$$ *[2]*

The angular version of Newton's second law then gives

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}.$$ *[1]*

Inserting the torque for the situation

$$\boldsymbol{\tau} = mga\cos\vartheta\hat{\boldsymbol{h}}$$ *[1]*

then completes the required expression.

(f) Given that the angular momentum may be written as $\mathbf{L} = I_d\omega_d\hat{\mathbf{d}}$, where ω_d the coin's instantaneous angular velocity, show that $\Omega\omega_d = 4g/a$. [3]

Substituting the given expression for $\mathbf{L}$ into the previous result,

$$\Omega I_d\omega_d\,\hat{\boldsymbol{\Omega}} \times \hat{\boldsymbol{d}} = mga\cos\vartheta\,\hat{\boldsymbol{h}} \qquad [1]$$

where evaluation of the vector product gives

$$\Omega I_d\omega_d\cos\vartheta\,\hat{\boldsymbol{h}} = mga\cos\vartheta\,\hat{\boldsymbol{h}}. \qquad [1]$$

Substituting the expression for the moment of inertia and cancelling common terms, we thus obtain

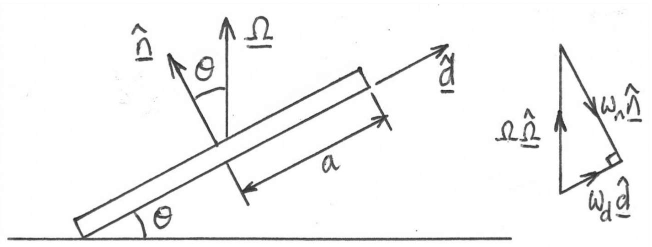
$$\Omega ma^2\omega_d/4 = mga$$

so that

$$\Omega\omega_d = 4g/a. \qquad [1]$$

(g) By considering the triangle of angular velocity vectors, or otherwise, show that the coin therefore precesses with an angular velocity

$$\Omega = \sqrt{\frac{4g}{a\sin\vartheta}}. \qquad [2]$$



From the triangle of angular velocity vectors, which illustrates that $\omega_d\hat{\mathbf{d}} = \Omega\hat{\boldsymbol{\Omega}} - \omega_n\hat{\boldsymbol{n}}$ as stated earlier, it is apparent that

$$\omega_d = \Omega\sin\vartheta. \qquad [1]$$

Substitution into the previous result gives

$$\Omega\omega_d = \Omega^2\sin\vartheta = 4g/a$$

and hence, as required,

$$\Omega = \sqrt{\frac{4g}{a\sin\vartheta}}. \qquad [1]$$

B2. (a) (i) Give an expression, in vector form, for the gravitational force $\mathbf{F}_{12}$ upon a body of mass m_1 at position $\mathbf{r}_1$, due to a second body of mass m_2 at position $\mathbf{r}_2$, in terms of the relative position $\mathbf{r}_{12} \equiv \mathbf{r}_1 - \mathbf{r}_2$. [2]

$$\mathbf{F}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_{12}|^2}\hat{\mathbf{r}}_{12} = -\frac{Gm_1m_2}{|\mathbf{r}_{12}|^3}\mathbf{r}_{12},$$

where G is the gravitational constant and $\hat{\mathbf{r}}_{12}$ is a unit vector in the direction $\mathbf{r}_{12}$.

[Either expression suffices for full marks.] *[2]*

(ii) Hence show that the gradient of the magnitude of the gravitational force has a radial component

$$\frac{\mathrm{d}F_{12}}{\mathrm{d}r_{12}} = -\frac{2Gm_1m_2}{r_{12}^3}.$$ [2]

Since the situation has spherical symmetry, we may reduce the vector expression to scalar form

$$F_{12} = \frac{Gm_1m_2}{r_{12}^2}.$$ *[1]*

and bear in mind that the force will be directed towards m_2. Differentiation then gives

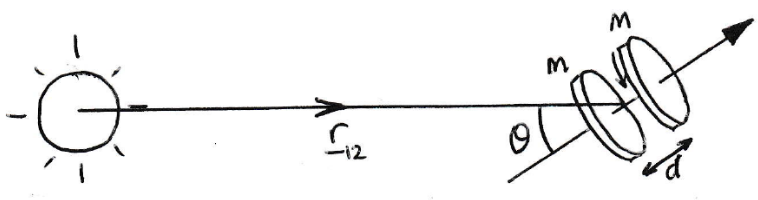
$$\frac{\mathrm{d}F_{12}}{\mathrm{d}r_{12}} = -\frac{2Gm_1m_2}{r_{12}^3}.$$ *[1]*

(b) A body in orbit about the Sun comprises two small discs, each of mass m, that are separated along their common axis by a distance d. The axis makes an angle ϑ to the plane of the orbit, and you may assume that $d \ll |\mathbf{r}_{12}|$, where $\mathbf{r}_{12}$ is the position of the body with respect to the Sun.

(i) Sketch the situation described, and show that the torque $\boldsymbol{\tau}$ acting upon the body will be

$$\boldsymbol{\tau} = \frac{GM_\odot md^2 \sin 2\vartheta}{2r_{12}^3},$$ [4]

where $r_{12} \equiv |\mathbf{r}_{12}|$ and $M_\odot$ is the mass of the Sun.



[1]

Since $d \ll |\mathbf{r}_{12}|$, we may take the gravitational forces $\mathbf{F}_a$, $\mathbf{F}_b$ acting upon the two discs to be approximately parallel. They differ however in magnitude because of the gravitational gradient, and hence have strengths

$$F_{a,b} = F_0 \pm \frac{d}{2}\cos\vartheta\frac{\mathrm{d}F_{12}}{\mathrm{d}r_{12}} = F_0 \mp \frac{2GmM_\odot d}{2r_{12}^3}\cos\vartheta,$$ *[1]*

where F_0 is the mean gravitational force experienced. The torque τ is then the moment of these forces about the centre of mass of the body,

$$\begin{aligned}\tau &= F_b\frac{d\sin\vartheta}{2} + F_a\frac{-d\sin\vartheta}{2} = \frac{d\sin\vartheta}{2}(F_b - F_a) \\ &= \frac{GmM_\odot d^2}{r_{12}^3}\cos\vartheta\sin\vartheta = \frac{GM_\odot md^2\sin 2\vartheta}{2r_{12}^3}.\end{aligned}$$

[2]

(ii) Hence show that, if the body has a moment of inertia I about the discs' common axis, and rotates about that axis with an angular velocity ω, its angular momentum vector will rotate with an angular velocity

$$\Omega = \frac{\tau}{I\omega\cos\vartheta},$$

and therefore that

$$\Omega = \frac{1}{\omega}\frac{GM_\odot}{r_{12}^3}\frac{md^2}{I}\sin\vartheta.$$ [6]

You may assume that Ω is small in comparison with both ω and the orbital angular velocity of the body about the Sun.

The effect of applying a torque τ to a rotating body with angular momentum $\mathbf{L}$ is described by the equation

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\tau}.$$ *[1]*

We see from the geometry of the situation here that the torque is always perpendicular to the angular momentum, and therefore cannot change its magnitude. The effect of the torque is therefore to rotate the rotation axis. *[1]*

For a vector rotating with angular velocity $|\boldsymbol{\Omega}|$ about an axis $\hat{\boldsymbol{\Omega}}$, where $\boldsymbol{\Omega} = |\boldsymbol{\Omega}|\hat{\boldsymbol{\Omega}}$,

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \boldsymbol{\Omega}\times\mathbf{L}.$$ *[1]*

Combining these expressions, we obtain the scalar relation

$$\tau = \Omega L\sin\alpha,$$

where $L = I\omega$ and α is the angle between $\boldsymbol{\Omega}$ and $\mathbf{L}$. *[1]*

TURN OVER

Students may know or assume that the precession axis is parallel to the orbital axis; this may be determined explicitly by noting that only this component of the precession will not average to zero as the body orbits the Sun. It follows that $\sin\alpha = \cos\vartheta$. We hence find

$$\Omega = \frac{\tau}{L\sin\alpha} = \frac{\tau}{I\omega\cos\vartheta}, \qquad [1]$$

and hence, substituting our previous result for $\boldsymbol{\tau}$,

$$\Omega = \frac{GmM_\odot d^2}{r_{12}^3 I\omega\cos\vartheta}\cos\vartheta\sin\vartheta = \frac{1}{\omega}\frac{GM_\odot}{r_{12}^3}\frac{md^2}{I}\sin\vartheta. \qquad [1]$$

(iii) Indicate, with the aid of a sketch, the direction and path of this precession with respect to the body, its rotation axis, and the Sun. [1]

[1]

(c) The Earth has a greater radius in its equatorial plane than along its polar axis. It may be modelled as a sphere of radius r_e, from which two discs of radius $\sqrt{2/5}\,r_e$, thickness $(r_e - r_p)$ and separation $2r_e$ have been subtracted, where $r_e = 6378$ km and $r_p = 6357$ km are the radii in the equatorial and polar directions. The density of the Earth near its surface is around 2750 kg m^{-3}, and the mean density of the Earth is 5514 kg m^{-3}. The mass of the Sun, $M_\odot = 1.989 \times 10^{30}$ kg, the distance of the Earth from the Sun is around 1.5×10^8 km, the gravitational constant $G = 6.674 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$, and the Earth's axis makes an angle $\vartheta = 66.6^\circ$ with its orbital plane.

(i) Using the data given, calculate the mass of the Earth, $M_\oplus$. [2]

Using the usual expression for the volume of the sphere, and the mean density ρ,

$$M_\oplus = \frac{4}{3}\pi r_e^3\rho = \frac{4}{3}\pi\,(6.378\times10^6\mathrm{m})^3\,5514\,\mathrm{kg\,m^{-3}} = 5.99\times10^{24}\,\mathrm{kg}. \qquad [2]$$

(The mass in the discs is about 1000 times smaller and hence negligible to this precision.)

(ii) Assuming that the moment of inertia of the Earth is approximately given by $(2/5)M_\oplus r_e^2$, find the period with which the Earth's axis precesses, at the time of the summer solstice, due to the effect of the Sun's gravity gradient upon the non-symmetrical Earth. [3]

Substituting given and derived values into the earlier expression,

$$\begin{aligned}\Omega &= \frac{1}{\omega}\frac{GM_\odot}{r_{12}^3}\frac{md^2}{I}\sin\vartheta \\ &= \frac{3600\times 24\,\mathrm{s}}{2\pi}\frac{6.674\times 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}\,1.989\times 10^{30}\,\mathrm{kg}}{(1.5\times 10^{11}\,\mathrm{m})^3}\times\sin(66.6^\circ)\times \\ &\quad \frac{2750\,\mathrm{kg\,m^{-3}}\,\pi(2/5)(6378\times 10^3\,\mathrm{m})^2\times(6378-6357)\times 10^3\,\mathrm{m}\times(2\times 6378\times 10^3\,\mathrm{m})^2}{(2/5)\times 5.99\times 10^{24}\,\mathrm{kg}\times(6378\times 10^3\,\mathrm{m})^2} \\ &= 2.45\times 10^{-12}\,\mathrm{rad\,s^{-1}}.\end{aligned}$$

[2]

The precession period is hence

$$\frac{2\pi}{\Omega} = 2.57\times 10^{12}\,\mathrm{s} = 81,400\,\mathrm{years}.$$ *[1]*

[This calculation underestimates the period by a factor of 2, as the torque varies around the orbit between the calculated value at the solstices and zero at the equinoxes. The precession of the Earth's rotation axis is in practice more due to the effect of the Moon, which generates a torque a little over twice that from the Sun. The density variation assumed here is possibly an over-estimate, the moment of inertia of the Earth is a factor of 1.2 greater than its observed value, and the factor of $\sqrt{2/5}$ is not quite accurate but allows the factor of $(2/5)$ in the Earth's moment of inertia to be cancelled. The ellipticity of the Earth's orbit also has a small effect. In reality, the mass distribution within the Earth is deduced from the observed precession period of 25,772 years.]

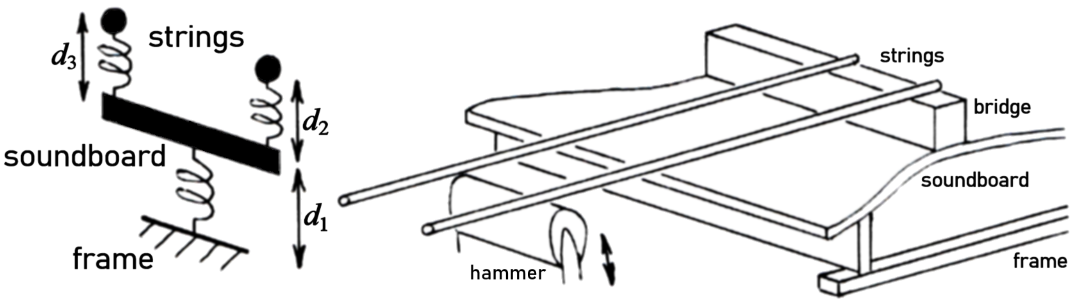
TURN OVER

B3. (a) Explain what is meant by (i) *simple harmonic motion* and (ii) the *normal mode* of an oscillating system. [4]

(i) Simple harmonic motion is that of a single body when subject to a restoring force that is proportional to its displacement, so that the displacement varies sinusoidally with time . *[2]*

(ii) A normal mode is a motion in which all parts of the system oscillate with the same single frequency and (therefore) with a fixed phase relationship between each other. *[2]*

Each mid-range note of a piano is produced by a pair of identical strings that pass over the same bridge to transmit their motions to the soundboard. The system can be modelled as a pair of equal masses m, representing the strings, that are connected through springs of natural length l and spring constant k, representing the restoring mechanisms of the displaced strings, to a mass m_0 corresponding to the soundboard, which is itself attached by a spring of natural length l_0 and constant k_0 to a solid anchor representing the piano frame. The effects of gravity and sideways or tilting motions can be neglected.



To analyse the piano dynamics, define the displacement of the soundboard from rest as $x_1 \equiv d_1 - l_0$, and $x_2 \equiv d_2 - l$ and $x_3 \equiv d_3 - l$ as the displacements of the string-soundboard distances from their rest values.

(b) Setting out your working formally, derive the three equations of motion

$$
\begin{aligned}
mm_0\ddot{x}_1 &= -k_0 m x_1 + km(x_2 + x_3) \\
mm_0\ddot{x}_2 &= k_0 m x_1 - k(m + m_0)x_2 - kmx_3 \\
mm_0\ddot{x}_3 &= k_0 m x_1 - kmx_2 - k(m + m_0)x_3
\end{aligned}
$$

where $\ddot{x}_1 \equiv \mathrm{d}^2 x_1/\mathrm{d}t^2$, etc. [4]

The acceleration of the soundboard will be $\ddot{x}_1$; the accelerations of the springs will be $(\ddot{x}_1 + \ddot{x}_2)$ and $(\ddot{x}_1 + \ddot{x}_3)$ respectively. *[1]*

The tension in a spring is the product of its extension and spring constant. If we label the soundboard 1 and the strings 2 and 3, the forces acting upon these three masses will thus be *[1]*

$$\begin{aligned} F_1 &= -k_0 x_1 + k(x_2 + x_3) \\ F_2 &= -k x_2 \\ F_3 &= -k x_3 \end{aligned}$$

Application of Newton's second law hence yields the equations of motion *[1]*

$$\begin{aligned} m_0 \ddot{x}_1 &= -k_0 x_1 + k(x_2 + x_3) \\ m(\ddot{x}_1 + \ddot{x}_2) &= -k x_2 \\ m(\ddot{x}_1 + \ddot{x}_3) &= -k x_3. \end{aligned}$$

Substitution of $\ddot{x}_1$ then gives, after multiplying through by m or m_0, *[1]*

$$\begin{aligned} m m_0 \ddot{x}_1 &= -k_0 m x_1 + k m (x_2 + x_3) \\ m m_0 \ddot{x}_2 &= -k m_0 x_2 + k_0 m x_1 - k m (x_2 + x_3) = k_0 m x_1 - k(m + m_0) x_2 - k m x_3 \\ m m_0 \ddot{x}_3 &= -k m_0 x_3 + k_0 m x_1 - k m (x_2 + x_3) = k_0 m x_1 - k m x_2 - k(m + m_0) x_3. \end{aligned}$$

(c) By substituting the normal mode solutions $x_j = a_j \exp(\mathrm{i}\omega t)$, where $j = 1 \ldots 3$, and assuming that $k_0 \ll k$, show that the common frequency of motion ω must satisfy

$$m m_0 \omega^2 \left(m m_0 \omega^2 - k m_0 \right) \left[m m_0 \omega^2 - k(2m + m_0) \right] = 0.$$ [7]

Substituting the given expressions $x_i = a_i \exp(\mathrm{i}\omega t)$ into the equations of motion [1], and cancelling the common factor $\exp(\mathrm{i}\omega t)$ [1], we obtain *[2]*

$$\begin{aligned} -\omega^2 m m_0 a_1 &= -k_0 m a_1 + k m (a_2 + a_3) \\ -\omega^2 m m_0 a_2 &= k_0 m a_1 - k(m + m_0) a_2 - k m a_3 \\ -\omega^2 m m_0 a_3 &= k_0 m a_1 - k m a_2 - k(m + m_0) a_3. \end{aligned}$$

which may be rearranged and written in matrix form

$$\begin{pmatrix} m m_0 \omega^2 - k_0 m & k m & k m \\ k_0 m & m m_0 \omega^2 - k(m + m_0) & -k m \\ k_0 m & -k m & m m_0 \omega^2 - k(m + m_0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0.$$ *[1]*

TURN OVER

Since $a_{1\ldots 3}$ cannot be obtained by operating on the zero vector, the matrix must be non-invertible, so its determinant must be zero. Omitting the terms in k_0 since $k_0 \ll k$, this gives

$$\begin{vmatrix} mm_0\omega^2 & km & km \\ 0 & mm_0\omega^2 - k(m+m_0) & -km \\ 0 & -km & mm_0\omega^2 - k(m+m_0) \end{vmatrix} = 0 \qquad [\,1\,]$$

which yields, from calculation of the determinant,

$$mm_0\omega^2 \left\{ \left[mm_0\omega^2 - k(m+m_0) \right]^2 - (km)^2 \right\} = 0. \qquad [\,1\,]$$

The term in curly brackets may be recognized as the difference of two squares, giving the factors directly; alternatively, it may be expanded to give

$$mm_0\omega^2 \left[\left(mm_0\omega^2\right)^2 - 2k(m+m_0)\left(mm_0\omega^2\right) + k^2 m_0 \left(2m+m_0\right) \right] = 0. \qquad [\,1\,]$$

The bracketted term can either be solved as a quadratic in $(mm_0\omega^2)$ or factorized by re-writing the above equation as

$$mm_0\omega^2 \left[\left(mm_0\omega^2\right)^2 - k\left[(2m+m_0)+m_0\right]\left(mm_0\omega^2\right) + k^2 m_0 \left(2m+m_0\right) \right] = 0.$$

from which the factors are immediately apparent, giving

$$mm_0\omega^2 \left[mm_0\omega^2 - km_0 \right] \left[mm_0\omega^2 - k(2m+m_0) \right] = 0. \qquad [\,1\,]$$

(d) Hence find expressions for the frequencies ω_{sym} and ω_{asym} of the *symmetric* and *asymmetric* modes, identifying which is which. [3]

Unless $mm_0 = 0$, the roots to the above equation are $\omega^2 = 0$, $\omega^2 = k(2m+m_0)/(mm_0)$ and $\omega^2 = k/m$. These correspond respectively to the stationary state, symmetric vibration and asymmetric vibration - that is, [2]

$$\begin{aligned} \omega_{\text{sym}} &= \sqrt{\frac{k(2m+m_0)}{(mm_0)}} \\ \omega_{\text{asym}} &= \sqrt{\frac{k}{m}}. \end{aligned}$$

In the asymmetric motion, the strings move in opposite motions, the soundboard does not move, and it is as if the strings were masses m each connected to a fixed point via a spring k. The asymmetric mode is identified by setting $(a_1\, a_2\, a_3) = (0, 1, -1)$ in the matrix equation, giving

$$mm_0\omega_{\text{asym}}^2 - k(m+m_0) + km = 0$$

and hence the value stated. [1]

(e) Describe and interpret the symmetric mode and, by considering the role of the soundboard in converting the string motion to sound, explain how the symmetric and asymmetric modes will compare in loudness and duration after the hammer has struck the strings. [2]

In the symmetric motion, the two strings act in unison; ω_{sym} may be re-written as $\sqrt{k/\rho}$, where $1/\rho \equiv 1/(2m) + 1/m_0$ is the reduced mass of the double-string and soundboard two-body system [1]. Since the soundboard moves, this mode is loud and short-lived; the asymmetric mode is in contrast quiet and sustained [1]. *[2]*

B4. A comet of mass m moves in the gravitational field of a star of mass M, and its position is described by its polar coordinates (r, ϑ) relative to the star. The gravitational potential is given by $\mathcal{V}(r) = GMm/r$. Assume that $M \gg m$.

(a) Show that the angular momentum of the comet about the star will be $L = mr^2\dot{\vartheta}$, where $\dot{\vartheta}$ signifies $\mathrm{d}\vartheta/\mathrm{d}t$, the rate of change of ϑ with time. [2]

The tangential (azimuthal) component of the velocity will be $r\dot{\vartheta}$ [1], so the angular momentum will be $L = mr\left(r\dot{\vartheta}\right) = mr^2\dot{\vartheta}$ [1]. *[2]*

(b) Show that the comet's total energy $\mathcal{E}$ may be written as

$$\mathcal{E} = \frac{m}{2}\dot{r}^2 + \left(\frac{L^2}{2mr^2} - \frac{GMm}{r}\right) \equiv \frac{m}{2}\dot{r}^2 + \mathcal{U}(r),$$

where $\dot{r} \equiv \mathrm{d}r/\mathrm{d}t$ and $\mathcal{U}(r)$ is the effective potential in which the comet's radial motion occurs. [4]

The kinetic energy may be written as [1]

$$\mathcal{T} = \frac{1}{2}mv^2 = \frac{m}{2}\left[\dot{r}^2 + \left(r\dot{\vartheta}\right)^2\right].$$

The second term may be written in terms of the conserved angular momentum L, eliminating $\dot{\vartheta}$ and giving [1]

$$\mathcal{T} = \frac{1}{2}mv^2 = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2}.$$

The total energy is the sum of the kinetic and potential energies [1]

$$\mathcal{E} = \mathcal{T} + \mathcal{V} = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + \mathcal{V}(r) = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$

The last two terms depend only upon the radial coordinate r and may hence be combined into an effective potential [1]

$$\mathcal{E} = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r} \equiv \frac{m}{2}\dot{r}^2 + U(r),$$

where

$$\mathcal{U}(r) \equiv \frac{L^2}{2mr^2} - \frac{GMm}{r}.$$ *[4]*

(c) Assuming that the comet follows an elliptical orbit with the star at one focus, show from these results that the length $2a$ of the ellipse's major axis will be

$$2a = \frac{GMm}{-\mathcal{E}}.$$ [4]

The equation of motion of the comet is

$$\mathcal{E} = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

where the total energy $\mathcal{E}$ and angular momentum L are conserved. At the aphelion and perihelion, $\dot{r} = 0$, so [1]

$$\mathcal{E} - \frac{L^2}{2mr^2} + \frac{GMm}{r} = 0,$$

which yields the quadratic equation [1]

$$\mathcal{E}r^2 + GMmr - \frac{L^2}{2m} = 0.$$

This has solutions [1]

$$r = \frac{-GMm \pm \sqrt{(GMm)^2 - 4\mathcal{E}\frac{L^2}{2m}}}{2\mathcal{E}}$$

The length of the major axis of the ellipse will be the sum of these two solutions, i.e. [1],

$$2a = \frac{GMm}{-\mathcal{E}}.$$ *[4]*

(d) By differentiating the total energy with respect to the time t, derive the equation of radial motion of the comet,

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{L^2}{m^2 r^3} - \frac{GM}{r^2}.$$ [2]

Differentiating the expression given in (b), where the total energy $\mathcal{E}$ and angular momentum L are conserved, we find [1]

$$0 = \frac{m}{2}2\dot{r}\ddot{r} - \frac{L^2}{mr^3}\dot{r} + \frac{GMm}{r^2}\dot{r}$$

whence, dividing through by $\dot{r}$ and rearranging, we obtain [1]

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} = \frac{L^2}{m^2 r^3} - \frac{GM}{r^2}.$$ *[2]*

TURN OVER

(e) By writing $\frac{\mathrm{d}}{\mathrm{d}t} \equiv \dot{\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta} \equiv \frac{L}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\vartheta}$ and making the substitution $r \equiv 1/u$, show that the equation of motion may be rewritten as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\vartheta^2} = -u + \frac{GMm^2}{L^2}.$$ [4]

Making the substitutions suggested, [1]

$$\frac{Lu^2}{m}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\frac{L}{m}u^2\frac{\mathrm{d}}{\mathrm{d}\vartheta}\frac{1}{u}\right) = \frac{L^2}{m^2}u^3 - GMu^2$$

which becomes [1]

$$\frac{L^2}{m^2}u^2\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(u^2\frac{-1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\vartheta}\right) = \frac{L^2}{m^2}u^3 - GMu^2$$

hence [1]

$$-\frac{L^2}{m^2}u^2\frac{\mathrm{d}^2 u}{\mathrm{d}\vartheta^2} = \frac{L^2}{m^2}u^3 - GMu^2$$

and thus [1]

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\vartheta^2} = -u + \frac{GMm^2}{L^2}.$$ *[4]*

(f) Hence show that the comet will trace out a path $r(\vartheta)$ of the form

$$r = \frac{L^2}{GMm^2\left(1+\alpha\cos\vartheta\right)}$$

where

$$\alpha^2 = 1 + \frac{2L^2\mathcal{E}}{\left(GMm\right)^2 m}.$$ [4]

The expression of part (e) defines simple harmonic motion of u about the point $u = GMm^2/L^2$. This may be shown explicitly by substituting the form given into the equation of motion [2]:

$$\frac{\mathrm{d}^2}{\mathrm{d}\vartheta^2}\frac{GMm^2\left(1+\alpha\cos\vartheta\right)}{L^2} = -\frac{GMm^2\left(\alpha\cos\vartheta\right)}{L^2} = -\frac{GMm^2\left(1+\alpha\cos\vartheta\right)}{L^2} + \frac{GMm^2}{L^2}.$$ *[2]*

Adding the maximum and minimum values of r we find the length of the major axis, and equate this to the result from part (c) [1]:

$$2a = \frac{L^2}{GMm^2}\frac{(1+\alpha)+(1-\alpha)}{(1+\alpha)(1-\alpha)} = \frac{L^2}{GMm^2}\frac{2}{1-\alpha^2} = \frac{GMm}{-\mathcal{E}}$$

hence [1]

$$\alpha^2 = 1 + \frac{2L^2\mathcal{E}}{\left(GMm\right)^2 m}.$$ *[2]*

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