WAVE PHYSICS exam January 2006.

A1. TRANSVERSE waves involve the propagation of a particle or medium displacement, or of a field component, in a direction perpendicular to the direction of wave propagation. [1 mark]

LONGITUDINAL waves involve the propagation of a displacement or field component parallel to the direction of propagation [1 mark]

Transverse waves: electromagnetic radiation, gravitational waves, guitar strings, shallow water waves [$\frac{1}{2}$ mark]

Longitudinal waves: sound, thermal waves [$\frac{1}{2}$ mark]

(Note that longitudinal components of e-m, gravitational and guitar waves are also possible, and that shallow water waves involve longitudinal motion in their propagation.)

Waves which are neither transverse nor longitudinal: quantum wavefunctions, combustion/flame wavefront, biochemical/neurological waves (eg. cephalopods), Mexican waves (perhaps), even waves of fear, .. [1 mark]

A2. $y = 0.005 \sin \pi(4x - 40t)$ with x, y in metres
t in seconds

is of the form $y = y_0 \sin(kx - \omega t)$ etc.

(a) amplitude $= 0.005$ m [$\frac{1}{2}$ mark]

wavelength $= 2\pi/k = 2\pi/4\pi = 0.5$ m [$\frac{1}{2}$ mark]

wavenumber $= k = 4\pi\ m^{-1}$ [$\frac{1}{2}$ mark]

[spectroscopists' wavenumber $\bar{\nu} = k/2\pi = 2\ m^{-1}$]

frequency $= \omega/2\pi = 40\pi/2\pi = 20$ Hz [$\frac{1}{2}$ mark]

period $= 1/\text{frequency} = 1/20 = 0.05$ s [$\frac{1}{2}$ mark]

phase velocity $= \omega/k = 40/4 = 10\ m.s^{-1}$ [$\frac{1}{2}$ mark]

(b) transverse speed $= \frac{\partial y}{\partial t} = (-40\pi)\,0.005 \cos(4\pi x - 40\pi t)$ [$\frac{1}{2}$ mark]

$\Rightarrow$ maximum transverse speed $= 40\pi\, 0.005 = 0.2\pi$ [$\frac{1}{2}$ mark]

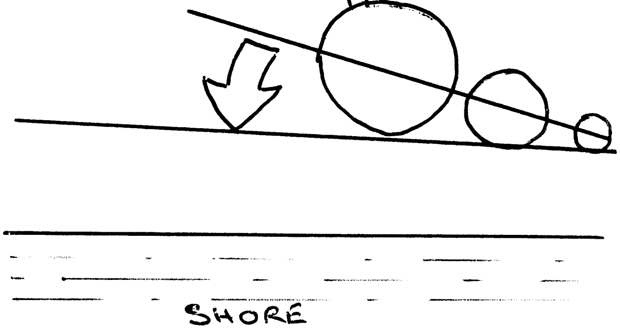
$\approx 0.63\ m.s^{-1}$

A3. Huygens' model of propagation:

each point on a wavefront acts as a secondary source
waves propagate from the secondary sources at the wave velocity in the medium
new wavefronts may then be constructed by finding the common tangent to adjacent secondary waves

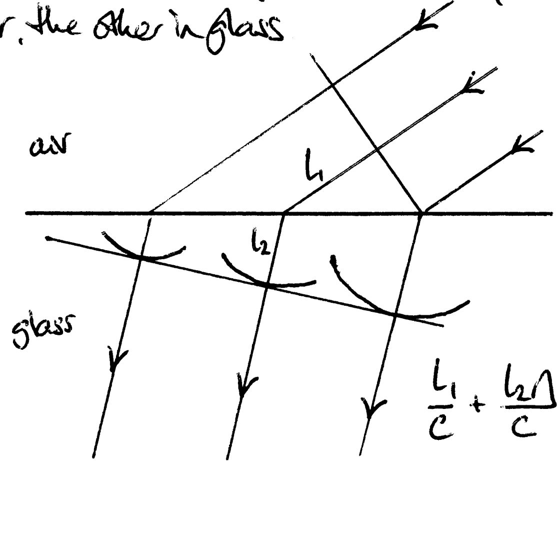
[2 marks]

(a) Wave speed $\sim h^{1/2}$ ⇒ waves slow as they enter shallow water
⇒ part of wave in deeper water travel faster + 'catch up' with the slower part already in shallow water
⇒ wave becomes more nearly parallel to shore.



[2 marks]

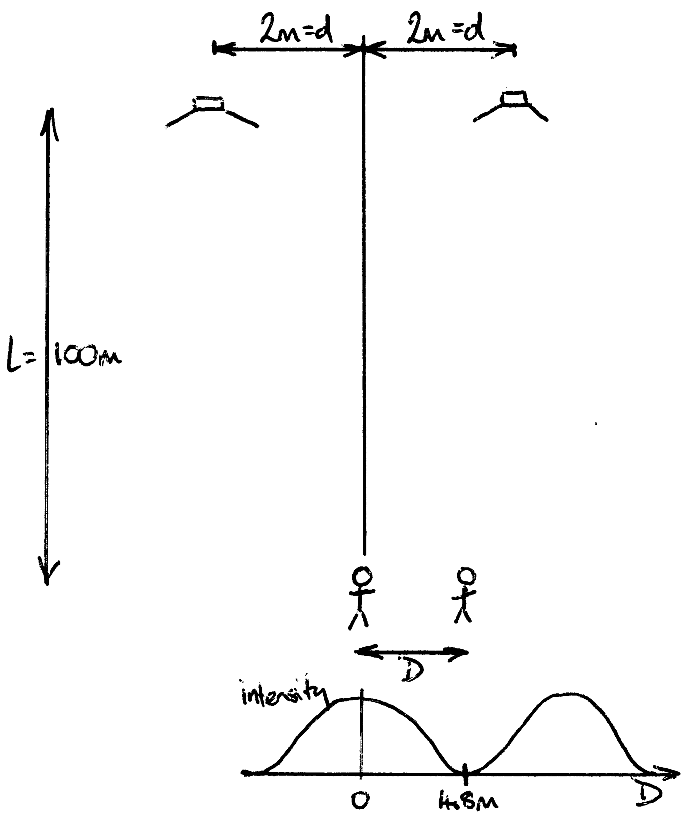
(b) The argument here is essentially the same as in (a), although complicated by the abrupt rather than gradual change in wave speed. We therefore break the time interval represented by the subsidiary wavefront into two parts: one in air, the other in glass



$$\frac{l_1}{c} + \frac{l_2 n}{c} = \Delta t$$

[2 marks]

A4. Interference: the addition of wave amplitudes, which may be of opposite sign (destructive interference) or similar sign (constructive interference). As different regions of a wave differ in amplitude and sign, the nature of the interference varies from position to position with the relative timing (phase) of the interfering components. [1 mark]



[1 mark]

Distance of observer from loudspeakers $= \sqrt{L^2 + (D \pm d)^2}$

$\Rightarrow$ path difference $= \sqrt{L^2 + (D+d)^2} - \sqrt{L^2 + (D-d)^2}$

which with $L = 100\,m$, $D = 4.8\,m$, $d = 2\,m$ gives $100.231 - 100.039 = 0.192\,m$

For the first minimum, this distance corresponds to a half wavelength

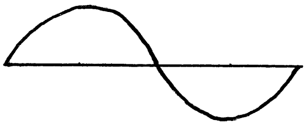
$\Rightarrow$ wavelength $= 0.192 \times 2\,m = \underline{\underline{0.383\,m.}}$ [2 marks]

(The small angle approximation is, to this precision, also valid.)

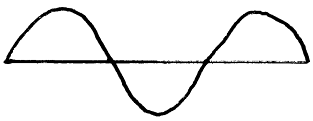
A5. Boundary conditions are constraints imposed upon the wave at particular positions by the presence of external influences. For example, a guitar string is constrained at the bridge and fret to have a fixed, zero displacement; the air column of a clarinet must have at its open end the unperturbed atmospheric pressure; shallow water waves at a harbour wall may move vertically but not horizontally. [2 marks]

A guitar string is fixed at both ends, and therefore supports sinusoidal standing waves in which the string length is an integer number of half-wavelengths. The corresponding frequencies are therefore integer multiples of the fundamental.

$L = \lambda/2 \quad f_0 = \frac{v}{\lambda} = v/2L$

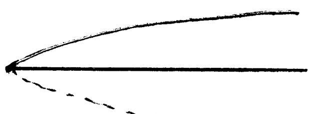


$L = 2\lambda/2 \quad f = 2\, v/2L = 2f_0$

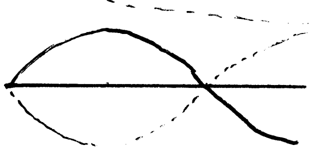


$L = 3\lambda/2 \quad f = 3\, v/2L = 3f_0$ [1 mark]

The clarinet is closed at the mouthpiece end (displacement = 0) and open to atmospheric pressure at the bell (pressure variation = 0). The pressure depends not upon the displacement but upon its spatial derivative – ie compression rather than simple movement of a given element of the air column. Nodes in the pressure variation hence coincide with anti-nodes of displacement.



$L = \lambda/4 \quad f_0 = \frac{v}{\lambda} = v/4L$



$L = 3\lambda/4 \quad f = 3v/4L = 3f_0$

[1 mark]

Hence the length = odd number of quarter wavelengths, and only odd harmonics are allowed.

B1. mass of element $= \rho\, \delta x$ [1 mark]

vertical components of tension $= T \sin\theta \approx T \tan\theta$ for small θ [1 mark]

$= T\, \partial y / \partial x$

$\Rightarrow$ net vertical component acting on element $= T \dfrac{\partial y}{\partial x}\left(x_0 + \dfrac{\delta x}{2}\right) - T \dfrac{\partial y}{\partial x}\left(x_0 - \dfrac{\delta x}{2}\right)$ [1 mark]

$\Rightarrow$ by Newton's law, [1 mark]

$$\rho\, \delta x\, \frac{\partial^2 y}{\partial t^2}(x_0) = T\left\{ \frac{\partial y}{\partial x}\left(x_0 + \frac{\delta x}{2}\right) - \frac{\partial y}{\partial x}\left(x_0 - \frac{\delta x}{2}\right)\right\}$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\, \frac{\frac{\partial y}{\partial x}\left(x_0 + \delta x/2\right) - \frac{\partial y}{\partial x}\left(x_0 - \delta x/2\right)}{\delta x}$$

which, taking the limit as $\delta x \to 0$, gives [1 mark]

$$\underline{\underline{\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}.}}$$

Substituting in the trial form given, for which

$$\frac{\partial^2 f}{\partial t^2} = (-\omega^2) f_0 \cos(kx - \omega t + \phi), \quad \frac{\partial^2 f}{\partial x^2} = (-k^2) f_0 \cos(kx - \omega t + \phi),$$ [2 marks]

$$-\omega^2 f_0 \cos(kx - \omega t + \phi) = \frac{T}{\rho}(-k^2) f_0 \cos(kx - \omega t + \phi)$$

$$\Rightarrow \quad \omega^2 = \frac{T}{\rho} k^2$$ [1 mark]

so trial form satisfies wave equation at all points and times provided that [1 mark]

$$\omega^2 = \frac{T}{\rho} k^2.$$

Here, k = wavenumber $(2\pi/\lambda)$

ω = angular frequency $(2\pi f)$

ϕ = phase of cosine function at $x=0, t=0$. [1 mark]

B1 cont'd. $\omega^2 = T/_\rho \, k^2 \Rightarrow \omega = \sqrt{\frac{T}{\rho}}\,k.$ (either form acceptable) [1 mark]

wave speed (phase velocity) given by $v_p = \omega/k$ [1 mark]

$\Rightarrow \; v = \sqrt{T/_\rho}$ [1 mark]

Where string is fixed, $y = 0$. [1 mark]

Hence if string has ends at $x=0$, $x=L$, $y(0,t) = y(L,t) = 0$.

Standing wave solutions are either found by re-solving the wave equations by separation of variables, or obtained by superposing travelling waves of opposite k, eg.

$y(x,t) = f_k(x,t) - f_{-k}(x,t)$ with $\phi = 0$ in each case to satisfy $y(0,t) = 0$.

$= \cos(kx - \omega t) - \cos(-kx - \omega t)$

$= \cos kx \cos\omega t + \sin kx \sin\omega t - \cos kx \cos\omega t + \sin kx \sin \omega t$

$= 2 \sin kx \sin\omega t$ [2 marks]

$\Rightarrow$ to solve $y(L,t) = 0$, $\sin kL = 0$

$\Rightarrow k = n\frac{\pi}{L}$. for integer n. [1 mark]

$\Rightarrow \omega = \sqrt{\frac{T}{\rho}}\,k = n\sqrt{\frac{T}{\rho}}\,\frac{\pi}{L}$

$\Rightarrow f = \frac{\omega}{2\pi} = n\sqrt{\frac{T}{\rho}}\,\frac{1}{2L}.$ [1 mark]

B1 cont'd. If $\rho = 0.37\,g.m^{-1} = 3.7\times10^{-4}\,kg.m^{-1}$

$L = 0.648m,$

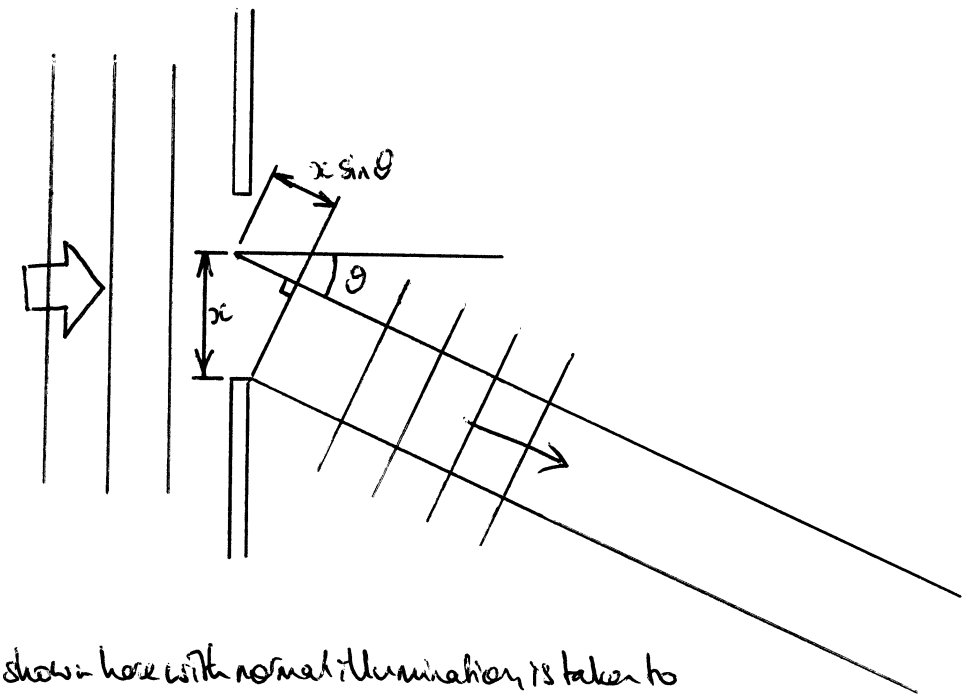
$f = 329.6\,Hz,$

then $T = \rho\left\{\frac{2Lf}{n}\right\}^2 = 3.7\times10^{-4}\left\{\frac{2\ 0.648\ 329.6}{1}\right\}^2$

$$= 67.5\,N.$$

[2 marks]

B2. Diffraction is the modification of a wave by its partial obstruction by an interposed object / mask. Fraunhofer diffraction is that observed sufficiently far from the mask that it may be considered a function of the observed angle alone. More specifically, Fraunhofer diffraction is observed when the optical path length from source to detector depends linearly upon the position coordinates within the diffracting mask. Equivalently, it is observed in (what would be, in the absence of the mask) the image plane of the source. [3 marks]



[2 marks]

If the slit, shown here with normal illumination is taken to extend from $x = -a/2$ to $x = a/2$ then the total diffracted amplitude will be

$$a(\theta) = a_0 \int_{-a/2}^{a/2} \exp i\left(\frac{2\pi}{\lambda} x \sin\theta\right) dx$$ [1 mark]

$$= \frac{a_0}{\pi \sin\theta/\lambda} \frac{1}{2i} \left[\exp i \frac{2\pi x \sin\theta}{\lambda}\right]_{-a/2}^{a/2}$$ [1 mark]

$$= \frac{a_0}{\pi \sin\theta/\lambda} \frac{e^{i\frac{\pi a \sin\theta}{\lambda}} - e^{-i\frac{\pi a \sin\theta}{\lambda}}}{2i}$$

B2. cont'd $\Rightarrow a(\theta) = \dfrac{a_0 a}{\pi a \sin\theta/\lambda} \sin \dfrac{\pi a \sin\theta}{\lambda}$

$$\sim \frac{\sin \frac{\pi a \sin\theta}{\lambda}}{\frac{\pi a \sin\theta}{\lambda}}.$$

[1 mark]

The convolution theorem is that the Fourier transform of the convolution of two functions is equal to the product of their individual transforms, ie.

$$FT\{A \circledast B\} = FT\{A\} \times FT\{B\}$$

Because of the symmetry between a Fourier transform and its inverse, it also follows that the Fourier transform of the product of two functions is equal to the convolution of their individual transforms

$$FT\{A \times B\} = FT\{A\} \circledast FT\{B\}.$$

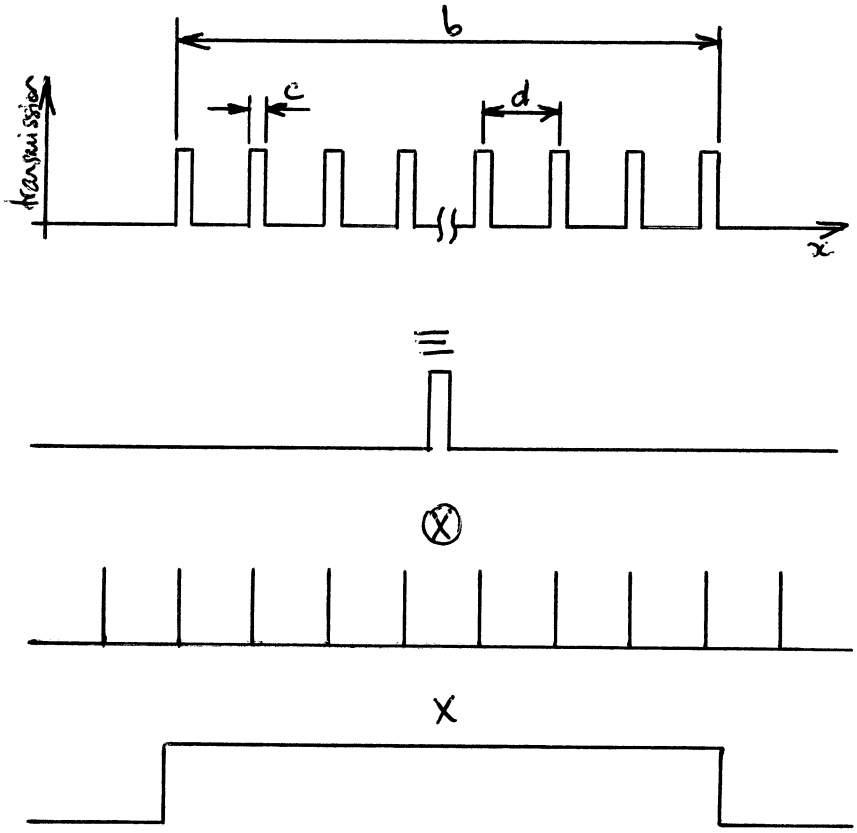
[2 marks]

Since the Fraunhofer diffraction pattern is related to the Fourier transform of the mask pattern, it follows that, if the latter may be broken down into products and convolutions of functions whose Fourier transforms (ie diffraction patterns) are known, then the overall diffraction pattern may be written without further detailed calculation.

[1 mark]

B2. cont'd.

(a)

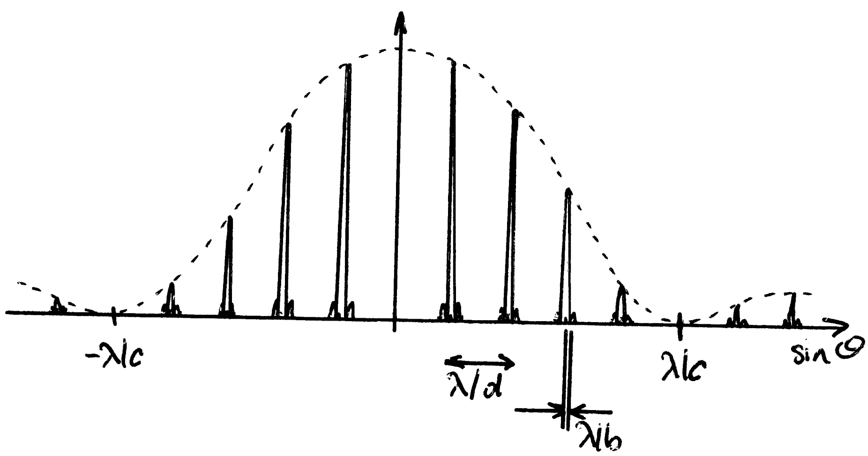


i.e. transmission of real grating ≡ (single slit ⊗ array of δ-f^{ns}) × wide slit [2 marks]

(b) The diffraction pattern will therefore be

{DP(single slit) × DP(array of δ-f^{ns})} ⊗ DP(wide slit) [2 marks]

(c)



[2 marks]

B2 cont'd. The angular width of each diffraction order is $\sim \delta\theta = \lambda / b$.
The diffraction angle (centre of order) depends upon wavelength through

$$\theta \sim \sin\theta = \lambda / d$$

Hence a change from θ to $\theta + \delta\theta$ corresponds to a change in wavelength of

$$\delta\lambda = \delta\theta \frac{d\lambda}{d\theta}$$ [1 mark]

$$= \delta\theta \, d$$

$$= \frac{\lambda d}{b}.$$ [1 mark]

With $\lambda = 500\text{nm}$
$d = \frac{1}{600}\text{ mm}$
$b = 50\text{mm}$,
this gives

$$\delta\lambda \sim 500\text{nm} \frac{1/600}{50} = \underline{\underline{0.017\text{nm}.}}$$ [1 mark]

B3. The Fourier transform allows a function of time or position to be instead represented as a function of frequency or spatial frequency – ie. by the spectrum of sinusoidal or complex exponential components into which it may be resolved. [2 marks]

The component with a given frequency is obtained by multiplying the function by a sinewave (or complex exponential wave) with the same (or negative) frequency, and integrating over the range of the function,

eg. $$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

$$f(\omega) = \int_{-\infty}^{\infty} f(t) \cos\omega t\, dt \quad \text{etc.}$$ [2 marks]

(The overall factor of $\frac{1}{\sqrt{2\pi}}$ is an arbitrary choice, determined by whether the aim is to symmetrize the Fourier transform and its inverse or to normalize the spectral intensity.)

The symmetry of $a(t)$ allows it to be expressed as a superposition of cosine waves $\cos\omega t$ of amplitudes $b(\omega)$, where

$$b(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(t) \cos\omega t\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} \cos\omega_0 t \cos\omega t\, dt$$ [1 mark]

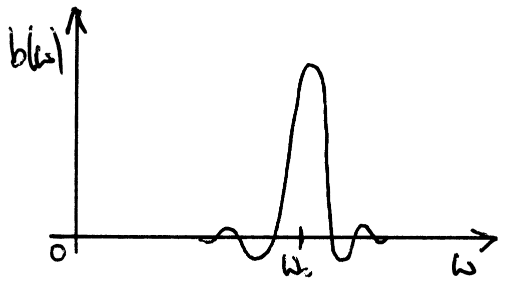
$$= \frac{1}{\sqrt{2\pi}} \int_{-T/2}^{T/2} \frac{1}{2}\left(\cos(\omega_0+\omega)t + \cos(\omega_0-\omega)t\right) dt$$ [1 mark]

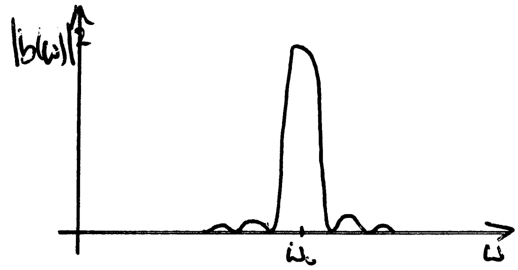
$$= \frac{1}{2\sqrt{2\pi}} \left[\frac{\sin(\omega_0+\omega)t}{\omega_0+\omega} + \frac{\sin(\omega_0-\omega)t}{\omega_0-\omega} \right]_{-T/2}^{T/2}$$ [1 mark]

B3 cont'd $\Rightarrow b(\omega) = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(\omega_0+\omega)^T/_2}{\omega_0+\omega} + \frac{\sin(\omega_0-\omega)^T/_2}{\omega_0-\omega}\right]$

Since $\omega_0 T \gg 1$, for frequencies around $\omega = \omega_0$ the second term dominates

$$\Rightarrow b(\omega) \approx \frac{T/2}{\sqrt{2\pi}} \frac{\sin(\omega_0-\omega)^T/_2}{(\omega_0-\omega)T/2}.$$ [1 mark]





[4 marks]

At half-maximum intensity, $|b(\omega)|^2 = \frac{1}{2}|b(\omega_0)|^2$ [1 mark]

$$\Rightarrow \left|\frac{\sin(\omega_0-\omega)^T/_2}{(\omega_0-\omega)T/2}\right|^2 = 1/2$$

i.e. $\frac{\sin^2 a}{a^2} = \frac{1}{2}$ where $a = \frac{\pm(\omega_0-\omega)T}{2}$

$$\Rightarrow \frac{(\omega_0-\omega)T}{2} = \pm 1.392$$

$$\Rightarrow |\omega_0-\omega| = \frac{2\ 1.392}{T}$$ [2 marks]

$$\Rightarrow \text{FWHM} = 2\,\frac{2\ 1.392}{T} = \frac{5.568}{T}\ \text{rads.sec}^{-1} \equiv \frac{0.886}{T}\ \text{Hz}.$$ [1 mark]

B3 cont'd. We'd expect the minimum distinguishable frequency difference to be of the same order as, or greater than, the FWHM, so we compare [1 mark]

$$\mathrm{FWHM}(T) = \frac{0.886\,\mathrm{Hz}}{T}$$

with the data plotted.

T	FWHM(T)	measured resolution
10 ms	89 Hz	~20Hz
20 ms	44 Hz	~ 7Hz
50 ms	18 Hz	~ 4Hz
100 ms	9 Hz	~ 3Hz
200 ms	4.4Hz	~ 2Hz
500 ms.	1.8 Hz.	~ 2Hz

[2 marks]

So, when the pulses are short, the ear can still distinguish separations that are significantly less than the FWHM. [1 mark]

B4. Dispersion is the variation of wave speed with frequency. For waves composed of several frequency components, it results in a change in the shape of the wave as it propagates. In a dispersive medium / system, the phase and group velocities will differ. [2 marks]

Dispersion is apparent in the variation of refractive index with wavelength, apparent in rainbows, chromatic aberrations in imaging systems and the dispersion of a spectrum by a prism. Pulses from a laser will become longer as they propagate down an optical fibre (unless waveguide effects compensate), and short electrical pulses will be distorted by electrical transmission lines. Localized quantum particles appear to exhibit a range of momentum values. [2 marks]

Dispersion by a prism can be used to produce an optical spectrum. The combination of dispersion with waveguide dispersion allows dispersion-free optical fibres to be designed. [1 mark]

Substituting the waveform given into the wave equation,

$$im \frac{\partial}{\partial t} f_0 \exp i(kx-\omega t+\phi) = -\frac{\partial^2}{\partial x^2} f_0 \exp i(kx-\omega t+\phi)$$ [1 mark]

$$\Rightarrow \quad im(-i\omega) f_0 \exp i(kx-\omega t+\phi) = -(ik)^2 f_0 \exp i(kx-\omega t+\phi)$$ [1 mark]

$$\Rightarrow \qquad m\omega = k^2$$

so the trial form given satisfies the wave equation for all times and positions provided that $m\omega = k^2$. [1 mark]

Dispersion relation

k is the wavenumber, the number of radians of phase per unit distance along the direction of propagation of the periodic wave. ω is the angular frequency, or phase per unit time

[2 marks]

B4 contd. The phase velocity is the velocity of a wavefront (point/line/surface of constant phase), and is the speed with which individual ripples of a group travel. The group velocity, which applies only to superpositions of single-frequency components, is the velocity with which the overall envelope moves. [2 marks]

The phase velocity is defined by $\underline{\underline{V_p = \omega/k}}$.

Here, $m\omega = k^2$, so
$$V_p = k/m = \sqrt{m\omega}/m = \underline{\underline{\sqrt{\omega/m}}}.$$ [1 mark]

$$f(x,t) = f_0 \exp i\left((k_0+\delta k)x - (\omega_0+\delta\omega)t + \phi_0\right) + f_0 \exp i\left((k_0-\delta k)x - (\omega_0-\delta\omega)t + \phi_1\right)$$ [1 mark]

$$= f_0 \exp i\left(k_0 x - \omega_0 t + \frac{\phi_0+\phi_1}{2}\right)\left\{\exp i\left(\delta k x - \delta\omega t + \frac{\phi_0-\phi_1}{2}\right) + \exp i\left(-\delta k x + \delta\omega t - \frac{\phi_0-\phi_1}{2}\right)\right\}$$ [1 mark]

$$= f_0 \exp i\left(k_0 x - \omega_0 t + \phi_A\right) 2\cos\left(\delta k x - \delta\omega t + \phi_B\right)$$ [1 mark]

where $\phi_A = \frac{\phi_0+\phi_1}{2}$, $\phi_B = \frac{\phi_0-\phi_1}{2}$

hence, with $f_1 = 2f_0$,

$$\underline{f(x,t) = f_1 \exp i\left(k_0 x - \omega_0 t + \phi_A\right) \underbrace{\cos\left(\delta k x - \delta\omega t + \phi_B\right)}_{\text{slowly varying modulation}}}.$$ [1 mark]

(Phases ϕ omitted from set question.)

B4 cont'd. $m\omega = k^2$

$$\Rightarrow (k_0+\delta k)^2 = m(\omega_0+\delta\omega)$$

$$(k_0-\delta k)^2 = m(\omega_0-\delta\omega)$$

$$\Rightarrow (k_0+\delta k)^2-(k_0-\delta k)^2 = m(\omega_0+\delta\omega)-m(\omega_0-\delta\omega)$$

$$\Rightarrow 4k_0\delta k = 2m\delta\omega$$

$\Rightarrow$ speed of modulation, $\frac{\delta\omega}{\delta k}$, $= 2k_0/m$.

The group velocity (which could equivalently be obtained by calculating $d\omega/dk$) is hence $\underline{\underline{2k_0/m = 2\sqrt{\omega/m}}}$.

This is twice the previously calculated phase velocity. [3 marks]