PHYS1015 MOTION AND RELATIVITY

JAN 2015 EXAM ANSWERS

Section A

A1. (Based on previously seen problem)

Displacement as function of time:

$$x(t) = A \sin \omega t$$

Frequency $f = \omega/2\pi$.

Velocity of mass is

$$v(t) = \frac{dx}{dt} = A\omega \cos \omega t$$

Maximal velocity is therefore $A\omega$ and is obtained when total energy E is equal to kinetic energy,

$$E = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2 .$$

[2]

Alternatively for the 2 marks: recall that $E = \frac{1}{2}kA^2$ for total energy and that $k = m\omega^2$ for oscillating spring.

Hence

$$A^2 = \frac{2E}{m\omega^2} = \frac{2E}{m(2\pi f)^2}$$

or

$$A = \frac{1}{2\pi f}\sqrt{\frac{2E}{m}} .$$

[1]

Putting in values we find

$$A = 0.16\,\mathrm{m}\,.$$

[1]

A2. (Similar to example given in workshops)

Weight of skydiver is mg. [1]

Forces balance at terminal velocity, since there is no acceleration.

Hence drag force also has magnitude mg and is opposite to direction of travel. [1]

Work done by drag force over distance d is therefore $-mgd$.

Power is rate of doing the work hence $P = -mgv$. [1]

Putting in numbers we find $P = -37$ kW. [1]

A3. (Unseen problem)

Relativistic energy $E = \gamma mc^2$, [1]

where $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. [1]

Rest energy is mc^2 and kinetic energy is $E - mc^2 = 0.99$ MeV. [1]

We have $\gamma = E/(mc^2)$ so

$$\frac{1}{1-\frac{v^2}{c^2}} = \frac{E^2}{m^2c^4} \quad \text{hence} \quad \frac{v^2}{c^2} = 1 - \frac{m^2c^4}{E^2}$$

and finally

$$\frac{v}{c} = \sqrt{1 - \frac{m^2c^4}{E^2}} = 0.94$$

[1]

A4. (Bookwork)

Magnitude of force on object at radius r is

$$\frac{GMm}{r^2}.$$

[1]

Work done by gravity moving mass from position r_0 to position r is

$$W = -\int_{r_0}^{r} dr' \frac{GMm}{(r')^2} = GMm\left[\frac{1}{r'}\right]_{r_0}^{r}.$$

[1]

and potential energy is defined to be $U(r) - U(r_0) = -W$. [1]

Taking reference point r_0 to be at infinite separation and declaring $U(\infty) = 0$ we have

$$U(r) = -\frac{GMm}{r}.$$

[1]

A5. (Based on previously seen problem in lectures)

Relativistic Doppler formula:

$$\nu' = \nu_0 \sqrt{\frac{1 - v_e/c}{1 + v_e/c}}$$

where ν_0 and ν' are emitted and observed frequencies. Hence for wavelengths:

$$\frac{\lambda'}{\lambda_0} = \sqrt{\frac{1 + v_e/c}{1 - v_e/c}}$$

[1]

Hence

$$\lambda_0^2(1 + v_e/c) = (\lambda')^2(1 - v_e/c)$$

[1]

So that

$$(v_e/c)(\lambda_0^2 + (\lambda')^2) = (\lambda')^2 - \lambda_0^2$$

[1]

or

$$(v_e/c) = \frac{(\lambda')^2 - \lambda_0^2}{\lambda_0^2 + (\lambda')^2} \approx 0.09$$

[1]

Section B

B1

(a) **(bookwork)**

In an elastic collision the total kinetic energy is conserved (Or: relative velocity after collision is opposite to that before). [1]

In an inelastic collision the total kinetic energy is not conserved. [1]

After a totally inelastic collision the two bodies move with common velocity. [1]

(b) **(similar to problem in lectures)**

(i) Conservation of momentum: $m_1 u = m_1 v_1 + m_2 v_2$,

where v_1 and v_2 are the velocities of the two balls in the positive x direction. [1]

Conservation of kinetic energy: $\frac{1}{2}m_1 u^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$. [1]

Rearranging KE equation: $m_1(u^2 - v_1^2) = m_2 v_2^2$,

hence $m_1(u - v_1)(u + v_1) = m_2 v_2^2$. [1]

Rearranging momentum equation: $m_1(u - v_1) = m_2 v_2$. [1]

Divide to get $u + v_1 = v_2$. [1]

Put back in to momentum equation: $m_1(u - v_1) = m_2(u + v_1)$.

Solve for v_1: i.e. $v_1(m_1 + m_2) = u(m_1 - m_2)$

so $v_1 = u\left(\frac{m_1 - m_2}{m_1 + m_2}\right)$. [1]

Moves in positive x direction if $m_1 > m_2$ and in negative x direction if $m_1 < m_2$. [1]

(ii) We had $v_2 = u + v_1$. But we know v_1 in terms of the initial velocity and the masses. So substitute in:

We have: $v_2 = u\left(1 + \frac{m_1 - m_2}{m_1 + m_2}\right) = 2u\frac{m_1}{m_1 + m_2}$ [1]

Kinetic energy of Ball 2 after collision is $\frac{1}{2}m_2 v_2^2 = 2\frac{u^2 m_1^2 m_2}{(m_1 + m_2)^2}$. [1]

(iii) Final height up incline z reached when KE is converted to potential energy $U = m_2 g z$. [1]

So we need $m_2 g z = 2\frac{u^2 m_1^2 m_2}{(m_1+m_2)^2}$. [1]

Hence $z = \frac{1}{2}\frac{v_2^2}{g} = \frac{2u^2 m_1^2}{g(m_1+m_2)^2}$. [1]

(c) **(Unseen problem)**

Totally inelastic means the balls move with common velocity after collision. [1]

Momentum conservation: $m_1 u = (m_1 + m_2)v$ for common velocity v afterwards. [1]

Hence $v = \frac{u m_1}{m_1+m_2}$.

Potential energy of combined ball at top of incline is $(m_1 + m_2)gh$. [1]

We need kinetic energy just after collision equal to (or greater than) this.

i.e. $\frac{1}{2}(m_1 + m_2)v^2 = (m_1 + m_2)gh$, i.e. $v^2 = 2gh$ [1]

i.e $\frac{u^2 m_1^2}{(m_1+m_2)^2} = 2gh$

i.e. $u = (1 + m_2/m_1)\sqrt{2gh}$. [1]

B2

(a) **(bookwork)**

Newton's second Law: $\mathbf{F} = m\mathbf{a}$. [1]

Gravitational force on projectile is $\mathbf{F} = -mg\hat{\mathbf{k}}$ with $\hat{\mathbf{k}}$ a vertical unit vector. [1]

Hence acceleration $\mathbf{a} = -g\hat{\mathbf{k}}$. [1]

Integrating once we find velocity $\mathbf{v} = \mathbf{v}_0 - gt\hat{\mathbf{k}}$. [1]

Integrating again we find position $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t - \frac{1}{2}gt^2\hat{\mathbf{k}}$. [1]

We can set initial position $\mathbf{r}_0 = 0$. [1]

Projecting onto horizontal component $x(t) = v_0 t\cos\theta$. [1]

and vertical component $z(t) = v_0 t\sin\theta - \frac{1}{2}gt^2$. [1]

(b) **(bookwork)**

Time of flight is time when height vanishes (but not $t = 0$), hence $v_0\sin\theta = \frac{1}{2}gt$. [1]

Rearranging we have $t = \frac{2v_0\sin\theta}{g}$. [1]

(c) **(Unseen problem)**

(i) Maximum height obtained at half the total time of flight, i.e. at $t = \frac{v_0\sin\theta}{g}$. [1]

At this time the height is

$$z_{\max} = \frac{v_0^2\sin^2\theta}{g} - \frac{1}{2}\frac{v_0^2\sin^2\theta}{g} = \frac{1}{2}\frac{v_0^2\sin^2\theta}{g}\,.$$

[1]

Hence (using $\sin(\pi/6) = 1/2$) we have $v_0^2 = 8gz_{\max}$ or $v_0 = \sqrt{8gz_{\max}} \approx 100\,\mathrm{m/s}$. [1]

Total distance travelled is value of x at total time of flight, i.e.

$$x_{\max} = \frac{2v_0^2\sin\theta\cos\theta}{g} = \frac{v_0^2\sin 2\theta}{g}\,.$$

[1]

Hence we have $x_{\max} = 8z_{\max}\sin 2\theta = 4\sqrt{3}z_{\max} = 60\sqrt{3}\,\mathrm{m} \approx 901\,\mathrm{m}$. [1]

(ii) Now elevation angle $\theta = 0$. So we can use simplified equations for x and z from before:

$x(t) = v_0 t$.

and

$z(t) = -\frac{1}{2}gt^2$ [1]

We want the distance travelled when $z(t) = -z_{\text{max}}$ with z_{max} the same as before. [1]

When $z(t) = -z_{\text{max}}$ we have

$t = \sqrt{\frac{2z_{\text{max}}}{g}}$. [1]

and hence horizontal distance travelled $x = v_0 \sqrt{\frac{2z_{\text{max}}}{g}}$. [1]

Putting in numbers:

$x = 520$ m. [1]

B3 (unseen problem)

(a)

(i) Let S be the laboratory reference frame and S' the electron beam frame.

S' moves in the direction of the electrons at $v = 0.995c$. [1]

The muon velocity in frame S is $u = 0.9c$.

For the muon in the laboratory reference frame we have:

$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 2.3.$ [1]

Thus its momentum in the S frame is $p = \gamma m u$ [1]

Hence$p = (2.3)(105.7\,\mathrm{MeV}/c^2)(0.9c) = 218\,\mathrm{Mev}/c$ [1]

(ii) To find the momentum in the electron-beam reference frame we use the velocity transformation equation to find the muon's velocity in frame S':

$u' = \frac{u-v}{1-\frac{uv}{c^2}} = \frac{0.99c-0.995c}{1-(0.9c)(0.995c)/c^2} = -0.91c$ [2]

In the laboratory frame the faster electrons are overtaking the slower muon. Hence the muon's velocity in the electron-beam frame is negative.

γ' for the muon in frame S' is:

$\gamma' = \frac{1}{\sqrt{1-0.91^2}} = 2.41$ [1]

The muon's momentum in the electron-beam reference frame is

$p' = \gamma' m u' = (2.41)(105.7\,\mathrm{MeV}/c^2)(-0.91c) = -231\,\mathrm{MeV}/c$ [1]

(b)

(i) In the CM frame both incoming protons have equal & opposite velocities and therefore momenta i.e. the total momentum is zero. [2]

Relativistic energy E of one of the protons is $\gamma m_p c^2$. Total energy is twice this.

Energy conservation: $2E = 2\gamma m_p c^2 = 2m_p c^2 + m_\pi c^2$

hence energy of each of the protons is $E = m_p c^2 + \frac{1}{2} m_\pi c^2 \approx 1010\,\mathrm{MeV}$ [2]

Since $E = \gamma m_p c^2$, we have $\gamma = 1 + \frac{m_\pi c^2}{2m_p c^2} = 1 + \frac{135}{2\times 938} = 1.072$

and hence $v = 0.36c$ [2]

(ii) Now one of the protons is at rest. So the velocity of the moving proton is found by relativistically adding the velocity in the centre of mass frame ($0.36c$) to the relative velocity between frames (also $0.36c$):

$u' = \frac{u-v}{1-\frac{uv}{c^2}} = \frac{0.36c+0.36c}{1+0.36^2} = 0.64c$ [2]

We have $\gamma = 1/\sqrt{1-0.64^2} = 1.3$ [2]

And for kinetic energy we have:

$K = (\gamma - 1)m_p c^2 = 0.3 \times 938 = 280\,\mathrm{MeV}$ [2]

Alternatively for the 6 marks:

In new frame collision products are moving with velocity v. [1]

Momentum p' of moving proton before collision is constrained by momentum conservation $p' = \gamma(2m_p + m_\pi)v$. [1]

Energy of incoming proton E' constrained by invariance of $E^2 - p^2c^2$, i.e.

we have $E^2 - p^2c^2 = (E')^2 - (p')^2c^2$ [2]

and hence $(E')^2 = E^2 - p^2c^2 + (p')^2c^2 = E^2 - (\gamma m_p v)^2c^2 + (\gamma(2m_p + m_\pi)v)^2c^2$.

which gives $E' = 1218$ MeV. [1]

Kinetic energy $K = E - m_pc^2 = 1218 - 938 = 280$ MeV. [1]

B4

(a)**(Bookwork)**

Postulate 1. The laws of physics are the same in any inertial frame. [1]

Postulate 2. The speed of light is the same in any inertial frame. [1]

An inertial frame is one in which Newton's first law holds, i.e. a body continues in uniform motion unless acted upon by an external force. [2]

(b)**(Bookwork)**

Lorentz transformation:

Orthogonal directions: $y' = y$ and $z' = z$ [1]

$t' = \gamma(t - vx/c^2)$ [2]

$x' = \gamma(x - vt)$ [2]

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ [1]

(c) **(Unseen problem)**

Let the first event occur at $x = t = 0$ in S (and hence at $x' = t' = 0$ in S'). [1]

The second event occurs at $x = \Delta x$ in S and at $t' = 0$ in S'. [1]

Using Lorentz transformation above we then have $0 = \gamma(t - v\Delta x/c^2)$. [1]

Hence the two events are separated in time in S by time $t = v\Delta x/c^2$. [1]

(d) **(Unseen problem)**

Signals are simultaneous in S' and separated in S, as in part (c):

Earlier event is **A** at origin $x = t = 0$. [1]

Later event is **B** at $x = \Delta x$ and time $t = v\Delta x/c^2$ in S. [1]

Hence $v = tc^2/(\Delta x) = 90$ m/s $(= 3 \times 10^{-7}c)$. [1]

S' was moving in positive x direction hence lander is flying in direction from **A** towards **B**. [1]

Yes - a pilot travelling faster in the same direction would see **B** occur first. [2]

END OF PAPER