
SEMESTER 2 EXAMINATION 2013-2014

ELECTROMAGNETISM

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Any vector field $\underline{v}$ satisfies:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \underline{\nabla}(\underline{\nabla}.\underline{v}) - \nabla^2 \underline{v}$$

Gauss's divergence theorem:

$$\int\int_{S_{closed}} \underline{v}.d\underline{A} = \int\int\int_{V} \underline{\nabla}.\underline{v}\, dV.$$

Stokes' theorem:

$$\int\int_{S_{open}} (\underline{\nabla} \times \underline{v}).d\underline{A} = \oint_{C} \underline{v}.d\underline{L}.$$

Section A

A1. Explain what is meant by charge density ρ and volume current density $\vec{J}$. Calculate the charge density ρ at a point inside a cube of side 0.2 m containing an electric charge of 8 mC uniformly distributed through its volume. If the cube moves with a speed of 2 m/s in the positive x direction, calculate the volume current density $\underline{J}$ at a stationary point inside the moving cube. [4]

A2. Calculate the electrostatic potential difference across a parallel plate capacitor of capacitance 6μF charged to 3 mC. Derive a formula for the electrostatic energy stored in the capacitor, and evaluate it in this case. [4]

A3. Explain, with reference to Gauss's divergence theorem, why the non-existence of magnetic monopoles implies that $\underline{\nabla}.\underline{B} = 0$. If monopoles were discovered, how would the above result change (ignoring constants)? [4]

A4. Explain in words what Poynting's vector represents. Calculate Poynting's vector at a point P where the electric field is $\underline{E} = 0.03\underline{\hat{x}}$ V/m and the magnetic field is $\underline{B} = 0.01\underline{\hat{x}} + 0.04\underline{\hat{y}}$ T. [4]

A5. Consider a static electromagnetic field in which the vector potential is given by

$$\underline{A} = x^2 y \underline{\hat{x}} + y^2 \underline{\hat{y}} + xy\underline{\hat{z}}.$$

Calculate the corresponding magnetic field and current density. [4]

Section B

B1. (a) Starting from the differential form of Gauss's law derive the integral form. [4]

(b) Consider a charged sphere of radius R centred at the origin with the spherically symmetric positive charge density

$$\rho(r) = \rho_0 \frac{r^3}{R^3}$$

where ρ_0 is a constant and r is the radial coordinate.

(i) Find the charge dQ' contained in a spherical shell of radius $r' < R$ and infinitesimal thickness dr' and hence show that the charge contained inside the sphere as a function of the distance from the origin is given by

$$Q(r) = 4\pi\rho_0 \frac{r^6}{6R^3}.$$

[4]

(ii) Using Gauss's law in integral form, show that the magnitude of the electric field $E_r(r)$ in the radial direction $\underline{\hat{r}}$ inside the sphere is given by

$$E_r(r) = \frac{Q(r)}{4\pi\epsilon_0 r^2}.$$

[2]

(iii) Verify Gauss's law in differential form at a general point inside the sphere. [4]

(iv) Using integration, calculate the potential $V(r)$ inside the sphere as a function of $r \leq R$. [6]

Note the results for the case of spherical symmetry:

$$\underline{\nabla}.\underline{E}(r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r(r))$$

$$\underline{\nabla} = \underline{\hat{r}}\frac{\partial}{\partial r}$$

TURN OVER

B2. (a) The Biot-Savart law below gives the differential magnetic field at a point described by a displacement vector $\underline{r}$ due to a current element $I d\underline{L}$ at another point described by a displacement vector $\underline{R}'$:

$$d\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{I d\underline{L} \times \hat{\underline{r}}'}{r'^2}.$$

Indicate on a diagram the vectors $\underline{r}$, $\underline{R}'$ and $\underline{r}'$. Express $\hat{\underline{r}}'$ and r' in terms of $\underline{r}$ and $\underline{R}'$. [5]

Write down the Biot-Savart law for the differential magnetic field for the case of a volume current density $\underline{J}(\underline{R}')$ contained in a small volume element dV' located at a point described by a displacement vector $\underline{R}'$. Include the derivation of any result you use in your answer. [3]

(b) A thin plastic circular ring of radius r has a positive charge q uniformly distributed along its circumference and rotates in the horizontal plane in an anti-clockwise direction as viewed from above at angular frequency ω. With the aid of a labelled diagram in each case, calculate the magnitude and direction of the magnetic field produced:

(i) at the centre of the ring? [4]

(ii) at a point on the axis of the ring at a distance d directly above the centre of the ring? [6]

Find approximate forms of the result in case (ii) for $d \ll r$ and $d \gg r$. [2]

B3. (a) Using the second derivative relation given on the front of the paper, show that Maxwell's equations in empty space imply that the electric field $\underline{E}$ and magnetic field $\underline{B}$ satisfy wave equations. [8]

(b) Show that the complex wave functions

$$\tilde{\underline{E}}(x, y, z, t) = \underline{E_0} e^{i(\underline{k}.\underline{r} - \omega t + \delta)}$$

$$\tilde{\underline{B}}(x, y, z, t) = \underline{B_0} e^{i(\underline{k}.\underline{r} - \omega t + \delta)}$$

satisfy wave equations of the same form as those in part (a). [4]

(c) Consider an electromagnetic wave travelling in the positive z direction, polarized in the x direction, through air (refractive index $n_1 = 1$), normally incident on glass (refractive index $n_2 = 1.5$) with the boundary at $z = 0$.

(i) Write down the complex E and B vector wave functions for the incident and reflected waves in air for $z < 0$, and for the transmitted wave in glass for $z > 0$. [3]

(ii) Write down the boundary conditions relating the transmitted electric field amplitude and phase to the incident and reflected electric field amplitudes and phases. [2]

(iii) Hence derive the formula for the reflection coefficient:

$$R = \frac{(n_2 - n_1)^2}{(n_2 + n_1)^2}$$

and evaluate it for light normally incident in air on glass. [3]

TURN OVER

B4. (a) Using Maxwell's equations for the divergence of the magnetic field and the curl of the electric field, show that the magnetic and electric fields can be written in terms of a magnetic vector potential $\underline{A}$ and a scalar potential V as:

$$\underline{B} = \underline{\nabla} \times \underline{A}, \quad \underline{E} = -\underline{\nabla} V - \frac{\partial \underline{A}}{\partial t}.$$

[4]

(b) Show that the electric and magnetic fields are invariant under the gauge transformations of the potentials,

$$\underline{A} \to \underline{A} + \underline{\nabla}\phi, \quad V \to V - \frac{\partial \phi}{\partial t}$$

for any arbitrary function $\phi = \phi(\underline{r}, t)$. [4]

(c) Using Maxwell's equations corresponding to Gauss's law and the Ampère-Maxwell law, derive the equations satisfied by V and $\underline{A}$ in the presence of charges and currents, in the Lorentz gauge defined by:

$$\underline{\nabla}.\underline{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}.$$

[10]

Rewrite these two equations using the d'Alembertian operator. [2]

END OF PAPER

