SEMESTER 2 EXAMINATION 2012-2013

ELECTROMAGNETISM

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Any vector field $\underline{v}$ satisfies:

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{v}) = \underline{\nabla}(\underline{\nabla}.\underline{v}) - \nabla^2 \underline{v}.$$

Gauss's divergence theorem:

$$\int\int_{S_{closed}} \underline{v}.d\underline{A} = \int\int\int_V \underline{\nabla}.\underline{v}\, dV.$$

.

Stokes' theorem:

$$\int\int_{S_{open}} (\underline{\nabla} \times \underline{v}).d\underline{A} = \oint_C \underline{v}.d\underline{L}.$$

.

Section A

A1. Given a static electric potential $V(x, y, z) = xy^2 + yz^2$ in Cartesian co-ordinates, derive an expression for the static electric field. [2]

A2. State Faraday's law in terms of the electromotive force (EMF) $\mathcal{E}$ and the magnetic flux Φ_B. By expressing $\mathcal{E}$ and Φ_B in terms of the electric and magnetic fields, derive the differential form of Faraday's law in terms of these fields. [4]

A3. Write down Ampère's law in integral form including Maxwell's displacement current. Hence derive the differential form of this equation. [5]

A4. Given that the magnetic and electric fields can be written in terms of the magnetic vector potential $\underline{A}$ and the scalar potential V as,

$$\underline{B} = \underline{\nabla} \times \underline{A}, \quad \underline{E} = -\underline{\nabla} V - \frac{\partial \underline{A}}{\partial t},$$

show that two of Maxwell's equations automatically follow. [4]

A5. Show that the differential form of Maxwell's equations in the vacuum of free space implies that the electric and magnetic fields satisfy wave equations. [5]

Section B

B1. (a) Starting from the integral form of Gauss's law derive the differential form. [4]

(b) Consider a charged sphere of radius R centred at the origin with the spherically symmetric positive charge density

$$\rho(r) = \rho_0 \left(1 - \frac{r^4}{R^4}\right)$$

where ρ_0 is a constant and r is the radial coordinate.

(i) Find the charge dQ' contained in a spherical shell of radius $r' < R$ and infinitesimal thickness dr'. [1]

Hence show that the charge contained inside the sphere as a function of the distance from the origin is given by

$$Q(r) = \frac{4}{3}\pi r^3 \rho_0 \left(1 - \frac{3}{7}\frac{r^4}{R^4}\right).$$

[3]

(ii) Using Gauss's law in integral form, show that the magnitude of the electric field $E_r(r)$ in the radial direction $\underline{\hat{r}}$ inside the sphere is given by

$$E_r(r) = \frac{Q(r)}{4\pi\epsilon_0 r^2}.$$

[3]

(iii) Verify Gauss's law in differential form at a general point inside the sphere. [3]

(iv) Using integration, calculate the potential $V(r)$ inside the sphere as a function of $r \leq R$. [6]

Note the results for the case of spherical symmetry:

$$\underline{\nabla}.\underline{E}(r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r(r))$$

$$\underline{\nabla} = \underline{\hat{r}}\frac{\partial}{\partial r}$$

TURN OVER

B2. (a) Draw a fully labelled diagram depicting a circular wire loop of radius R carrying a current I, where the loop is in the (x, y) plane, centred at the origin, with the current going anticlockwise. [1]

The differential magnetic field at the origin due to a current element $I d\underline{L}$ is given by the Biot-Savart Law,

$$d\underline{B} = \frac{\mu_0}{4\pi} \frac{I d\underline{L} \times \underline{\hat{r}}'}{r'^2}.$$

Draw the current element on your diagram, labelling the vectors $d\underline{L}$, $\underline{\hat{r}}'$, $d\underline{B}$ and the distance r'. [2]

Using symmetry arguments, show that the magnitude of the total magnetic field $\underline{B}$ at the origin is

$$\frac{\mu_0 I}{2R}.$$

[4]

(b) (i) If the circular wire loop in (a) carries a steadily increasing current

$$I = I_0 \frac{t}{t_0},$$

where t is time and I_0 and t_0 are constants, state the direction of the **curl** of the electric field $\underline{E}$ at the origin, and show that its magnitude is

$$|\underline{\nabla} \times \underline{E}| = \frac{\mu_0}{2R} \frac{I_0}{t_0}.$$

[3]

(ii) Draw a new diagram which depicts the electric field $\underline{E}$ and the Poynting vector $\underline{S}$ in the region of the (x, y) plane close to the origin. [2]

(iii) Using Stokes' theorem, hence estimate the magnitude of the electric field at a point close to the origin at a small radius $r \ll R$. Then estimate the Poynting vector at the point. Where does the energy originate from and how is it dissipated? [6]

(iv) Estimate the magnitude of the force on a test positive charge q moving radially away from the origin with speed v while at a distance r away from the origin. [2]

B3. (a) Consider an oscillating electric dipole with a charge $q(t)$ at the point $(0, 0, d/2)$ and $-q(t)$ at the point $(0, 0, -d/2)$ where the charges vary with time as $q(t) = q_0 \cos \omega t$ where q_0 and ω are constants. Show that the retarded potential at a point P is given by

$$\frac{q_0}{4\pi\epsilon_0}\left(\frac{\cos[\omega(t-\frac{r_1}{c})]}{r_1} - \frac{\cos[\omega(t-\frac{r_2}{c})]}{r_2}\right),$$

where r_1, r_2 are the respective distances of the charges $+q, -q$ to point P. [2]

Show, by expanding in small quantities, that the retarded potential at time t at the point $(0, 0, z)$, is given by approximately,

$$-\frac{q_0}{4\pi\epsilon_0}\frac{\omega d}{zc}\sin\left[\omega(t-\frac{z}{c})\right]$$

where $z \gg c/\omega \gg d$. [4]

(b) Suppose the oscillating electric dipole in part (a) gives rise to electric and magnetic fields at large distances $r \gg c/\omega \gg d$,

$$\underline{E} \approx -\frac{\mu_0\omega^2 q_0 d}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega(t-\frac{r}{c})\right]\underline{\hat{\theta}}$$

$$\underline{B} \approx -\frac{\mu_0\omega^2 q_0 d}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega(t-\frac{r}{c})\right]\underline{\hat{\phi}},$$

at a point P described by (r, θ, ϕ) in spherical polar coordinates.

Given the above results, explain fully why these fields describe spherical electromagnetic waves (your explanation should include a calculation of the Poynting vector). [5]

(c) Assuming a mobile phone to act as an oscillating electric dipole, estimate the magnitude of the peak electric field strength in Vm^{-1} at a base receiver 1 km away. In making your estimate, assume the mobile phone to operate at 100 MHz and to have a vertically held aerial 1 cm long carrying a peak current of 10 mA. (Note that $\mu_0 = 4\pi \times 10^{-7}\mathrm{TmA}^{-1}$.) [5]

Estimate the average power (in Watts per square metre) arriving at the base receiver from the mobile phone. [4]

TURN OVER

B4. (a) By applying Faraday's law to a very thin rectangular loop which straddles the plane boundary between two different dielectric media, with the aid of a diagram derive the approximate boundary condition for the parallel components of the electric fields on either side of the boundary. [4]

(b) Fresnel's equations for light polarized parallel to the incident plane are,

$$E_R e^{i\delta_R} = \frac{\alpha - \beta}{\alpha + \beta} E_I e^{i\delta_I}$$

$$E_T e^{i\delta_T} = \frac{2}{\alpha + \beta} E_I e^{i\delta_I}$$

where,

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}, \quad \beta = \frac{n_2}{n_1},$$

and I labels incident, R labels reflected and T labels transmitted electric field amplitudes, phases and ray angles in the plane of incidence.

(i) At noon, the Sun is directly overhead a flat sea. Find the reflected and transmitted electric field amplitude and phase in terms of the incident electric field amplitude and phase. Hence calculate the reflection and transmission coefficients. [6]
[Assume that sea water has refractive index $n_2 = 4/3$ and air has $n_1 = 1$.]

(ii) Later in the afternoon, a student observes the sunlight at Brewster's angle. Explain what is meant by Brewster's angle and why, at this time, the student finds Polaroid sunglasses to be particularly effective for reducing the reflected glare of the sunlight. [2]

(iii) With the aid of a diagram, explain why Brewster's angle corresponds to the condition $\theta_T + \theta_R = \pi/2$. Using this result, together with Fresnel's equations and the law of reflection, derive a formula for Brewster's angle. Estimate its value for sunlight reflecting off sea water. [6]

(iv) Using Fresnel's equations, show why the student finds the sunset to be particularly dazzling. [2]

END OF PAPER