
SEMESTER 1 EXAMINATION 2013 - 14

QUANTUM PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. Consider the probability density $\rho(x) = Ax^2(10 - x)$ if $0 \leq x \leq 10$ and $\rho(x) = 0$ otherwise. A is a constant. Determine A and the probability that a measurement of x will give a value greater than 5. [4]

A2. At time $t = 0$, a particle in one dimension has the wave function $A \exp[-x^2/a^2]$ where a and A are constants. Determine A in terms of a. [4]

You may find it helpful to note that

$$\int_0^\infty e^{-x^2/b^2}\, dx = \sqrt{\pi}\, \frac{b}{2},$$

where b is a real constant.

A3. The normalised energy eigenfunctions for a particle of mass m in the infinite square well with the potential $V(x) = 0$ if $0 \leq x \leq a$ and $V(x) = \infty$ otherwise are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right), \qquad n = 1, 2, 3 \cdots,$$

for $0 \leq x \leq a$ and $\psi_n(x) = 0$ otherwise. Determine the corresponding energy eigenvalues. [4]

A4. Starting with the expression $\vec{L} = \vec{r} \times \vec{p}$ for the angular momentum of a particle in terms of its position ($\vec{r}$) and momentum ($\vec{p}$), calculate the commutator $[L_x, L_z]$. Express your result in terms of the components of $\vec{L}$. [4]

A5. The spin operators $\vec{S} = (S_x, S_y, S_z)$ for a spin-$\frac{1}{2}$ particle can be represented by

$$S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Determine the eigenvalues and eigenvectors of S_x in this representation. [4]

Section B

B1. (i) The wave function for an electron in the ground state of a hydrogen atom is (in spherical polar coordinates)

$$\psi(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}\,,$$

where $a = 0.529 \times 10^{-10}$ m is the Bohr radius.

Find $\langle r \rangle$ for an electron in this state, expressing your answer in terms of the Bohr radius. [6]

You may find the following integral helpful:

$$\int_0^\infty r^n\, e^{-r/b}\, dr = n!\, b^{n+1}\,,$$

where n is a non-negative integer and b is a positive constant.

(ii) The energy eigenfunctions of an electron in the hydrogen atom are conventionally written as $\psi_{nlm}(\vec{r})$. Explain the significance of the three labels n, l, m. (You are not expected to evaluate any commutators explicitly.)

How many degenerate states are there when $n = 3$? (The ground state corresponds to $n = 1$ and you should neglect any additional degeneracies due to the spin of the electron.) [6]

(iii) The radial wave function for an electron with wave function $\psi_{210}(\vec{r})$ is proportional to $r\exp(-\frac{r}{2a})$. Without performing any integrals, explain whether you would expect $\langle r \rangle$ in this case to be smaller, equal or larger than that found in part (i). [3]

TURN OVER

(iv) An electron undergoes a transition from the state with wave function $\psi_{210}(\vec{r})$ to the ground state. Calculate the wavelength of the emitted photon.
(You may recall that the energy of an electron with wave function $\psi_{nlm}(\vec{r})$ is proportional to $1/n^2$ and that the ionisation energy of an electron in the ground state ($n = 1$) is 13.6 eV. 1 eV=1.602×10^{-19} J.) [5]

B2. Consider a particle in one-dimension with wave function $\Psi(x, t)$, where x and t represent its position and the time respectively. Let $\hat{Q}$ be the Hermitian operator representing the observable $Q(x, p)$, where p is the particle's momentum.

(i) Explain what is meant by a Hermitian operator. [4]

(ii) Show explicitly that the momentum operator (in one dimension) is Hermitian. [4]

(iii) Demonstrate that the eigenvalues of a Hermitian operator are real. [4]

(iv) Let $f(x)$ and $g(x)$ be two eigenfunctions of $\hat{Q}$ with different eigenvalues. Show that $f(x)$ and $g(x)$ are orthogonal. (You may assume that the spectrum of $\hat{Q}$ is discrete.) [4]

(v) Let $\hat{Q}'$ be the Hermitian operator representing a second observable $Q'(x, p)$. The generalised uncertainty principle states that

$$\sigma_Q^2 \sigma_{Q'}^2 \geq \left(\frac{1}{2i} \langle \left[\hat{Q}, \hat{Q}' \right] \rangle \right)^2$$

where $\left[\hat{Q}, \hat{Q}' \right]$ represents the commutator of $\hat{Q}$ and $\hat{Q}'$ and σ_Q and $\sigma_{Q'}$ represent the uncertainties on the corresponding observables, e.g.

$$\sigma_Q^2 = \langle \left(\hat{Q} - \langle Q \rangle \right) \Psi | \left(\hat{Q} - \langle Q \rangle \right) \Psi \rangle .$$

Using this result, deduce the Heisenberg uncertainty principle for position and momentum operators. [4]

TURN OVER

B3. (i) A free particle of mass m satisfies the one-dimensional time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi\,.$$

Show that

$$\psi(x) = Ae^{ikx}\,,$$

(where A is a constant) is a solution of this equation. Give the expression for k in terms of the quantities in the Schrödinger equation above. [4]

(ii) Explain why the solution above is not a physical one and explain what is meant by a wavepacket. [5]

(iii) At time $t = 0$ a free particle has a one-dimensional wave function $\Psi(x,t)$ with

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\,e^{ikx}\,dk\,,$$

where $\phi(k)$ is a given function of k. Explaining your reasoning, write down the integral expression for $\Psi(x,t)$ at a subsequent time t. [4]

(iv) Consider a general wave-packet with the form:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\,e^{i(kx-\omega(k)\,t)}\,dk$$

and assume that $\phi(k)$ is narrowly peaked at the value $k = k_0$. Explain the concept of the *group velocity* (v_{group}) and show that

$$v_{\mathrm{group}} = \frac{d\omega}{dk}\,.$$ [7]

B4. Consider the Hamiltonian of a particle of mass m moving in the x-direction in the potential of a Harmonic Oscillator,

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 ,$$

where ω is a constant.

(i) Given that the raising (a_+) and lowering (a_-) operators are given by

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp ip + m\omega x\right) ,$$

where p is the momentum operator, show that the commutator $[a_-, a_+] = 1$ and that $H = \hbar\omega\,(a_+a_- + \frac{1}{2}) = \hbar\omega\,(a_-a_+ - \frac{1}{2})$. [5]

(ii) Show that the energy eigenvalues are given by $E_n = (n + \frac{1}{2})\hbar\omega$, $(n = 0, 1, 2, \cdots)$. [5]

(iii) Determine the wave functions of the ground and first excited states. (You are <u>not</u> expected to normalise the wavefunctions.) [5]

(iv) By using the separation of variables (or otherwise) find the spectrum of the three-dimensional harmonic oscillator with Hamiltonian

$$H = -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega^2 r^2 ,$$

where $r^2 = x^2 + y^2 + z^2$. Determine the energies of the first three levels and their degeneracies. [5]

END OF PAPER