SEMESTER 1 EXAMINATION 2014-2015

QUANTUM PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. The wave-function of an electron in the ground state of a hydrogen atom takes the form

$$\psi(r,\theta,\phi) = Ae^{-r/a}\,,$$

where a and A are constants (a is real and positive) and (r,θ,ϕ) are spherical polar coordinates with the nucleus of the atom at the origin.

(i) What are the dimensions of the constants a and A? [2]

(ii) Determine A in terms of a. [3]

You may find it helpful to note that

$$\int_0^\infty x^n e^{-x/b}\,dx = n!\,b^{n+1}\,,$$

where b is a positive real constant and $n \geq 0$ is an integer.

A2. The normalised energy eigenfunctions for a particle of mass m in the infinite square well with the potential $V(x) = 0$ if $0 \leq x \leq a$ and $V(x) = \infty$ otherwise are

$$\psi_n(x) = \sqrt{\frac{2}{a}}\,\sin\left(\frac{n\pi}{a}\,x\right), \qquad n = 1, 2, 3\cdots\,,$$

for $0 \leq x \leq a$ and $\psi_n(x) = 0$ otherwise. Without performing any integrals, explain what is meant by the statement that the functions $\{\psi_n(x)\}$ are *orthonormal*. [3]

A3. In quantum mechanics, observables are represented by Hermitian operators. Explain what is meant by a Hermitian operator. [4]

A4. The spin operators $\vec{S} = (S_x, S_y, S_z)$ for a spin-$\frac{1}{2}$ particle can be represented by the matrices:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A particle is in the spin-state

$$\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(i) Verify that χ an eigenstate of S_x and determine the corresponding eigenvalue? [2]

(ii) A measurement is made of the z-component of the spin of a particle in state χ. What is the probability that the measurement will give the value $\hbar/2$. [2]

A5. A particle of mass m is in the ground-state of a one-dimensional harmonic oscillator with potential $\frac{1}{2}m\omega^2x^2$, where ω is a constant and x is the spatial coordinate. Given that the ground-state wave function is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{m\omega}{2\hbar}x^2\right],$$

calculate the expectation value of the potential energy of the particle. [4]

You may find it helpful to note that

$$\int_0^\infty dx\, x^{2n}\, e^{-x^2/\alpha^2} = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{\alpha}{2}\right)^{2n+1},$$

where α is a positive constant and n is a non-negative integer.

TURN OVER

Section B

B1. (i) Consider a particle in one dimension with wave function $\Psi(x,t)$ which satisfies the Schrödinger equation. Show that

$$\frac{d\langle x\rangle}{dt} = -\frac{i\hbar}{m}\int dx\,\Psi^*(x,t)\,\frac{\partial\Psi(x,t)}{\partial x}\,.$$

Using this result, derive the form of the momentum operator, p. [10]

You may find it helpful to note that

$$\frac{\partial^2\Psi^*}{\partial x^2}\Psi - \Psi^*\frac{\partial^2\Psi}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial\Psi^*}{\partial x}\Psi - \Psi^*\frac{\partial\Psi}{\partial x}\right]\,.$$

(ii) Given your result for the momentum operator in (a), derive the expression for the commutator of x with p. [4]

(iii) Consider light from a laser at a nominal wavelength of 800 nm and with pulses which are smaller than 30 μm in their direction of travel. State the Heisenberg uncertainly principle and estimate the uncertainty in the wavelength of a photon in the pulse. [6]

B2. Consider a particle moving in the x direction in the potential of a one-dimensional harmonic oscillator with units conveniently chosen so that the Hamiltonian operator $H = p^2 + x^2$, where p is the momentum operator.

(i) Show that

$$H = (x + ip)(x - ip) - \hbar\,.$$

[4]

(ii) Defining the raising and lowering operators $a_\pm$ by

$$a_\pm = \frac{1}{\sqrt{2\hbar}}\,(x \mp ip)$$

show that $[a_-, a_+] = 1$. [5]

(iii) By using the commutation relation in part ii), show that if $\psi(x)$ is a solution of the time-independent Schrödinger equation with energy E, then $a_+\psi(x)$ is a solution with energy $E + 2\hbar$. [5]

(iv) Write down the ground-state wave function of the particle (you may simply modify $\psi_0(x)$ given in question A5) and determine the wave function of the first excited state (you are not expected to determine the normalisation constant). [6]

TURN OVER

B3. (i) Why are the solutions of the Schrödinger equation corresponding to a free particle with a single energy unphysical?

Explain carefully what is meant by the terms *wavepacket* and *group velocity*. [6]

(ii) A free particle of mass m in one-dimension is initially localised in the range $-a \le x \le a$ with the wave function

$$\begin{aligned}\Psi(x, t=0) &= \frac{1}{\sqrt{2a}} \qquad \text{if } -a \le x \le a \\ &= 0 \qquad\qquad \text{otherwise}\,.\end{aligned}$$

Writing

$$\Psi(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \phi(k)\, e^{ikx}$$

determine $\phi(k)$ and hence show that at a general time t

$$\Psi(x, t) = \frac{1}{\pi\sqrt{2a}} \int_{-\infty}^{\infty} dk\, \frac{\sin(ka)}{k}\, e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}\,.$$

[8]

(iii) What is the group velocity of the wavepacket defined in part (ii).

Explaining your reasoning, state whether you would expect the wavepacket to disperse more quickly for small a or for large a. [6]

B4. (i) The energy eigenfunctions of an electron in the hydrogen atom are conventionally written as $\psi_{nlm}(\vec{r})$. Explain why the eigenfunctions can be labelled by n, l and m and what the significance of each of these labels is. (You are not expected to evaluate any commutators explicitly.)

How many degenerate states are there when $n = 2$?
(The ground state corresponds to $n = 1$ and you should neglect any additional degeneracies due to the spin of the electron.) [6]

(ii) For the state $n = 2, l = 1, m = 0$ the normalized wave function is (in spherical polar coordinates)

$$\psi_{210}(r, \theta, \phi) = \frac{1}{4\sqrt{2\pi a^3}} \cos(\theta) \frac{r}{a} e^{-\frac{r}{2a}} ,$$

where $a = 0.529 \times 10^{-10}$ m is the Bohr radius.
Find $\langle r \rangle$ for an electron in this state, expressing your answer in terms of the Bohr radius. [8]
You may find the integral given in question A1 to be helpful.

(iii) An electron undergoes a transition from the state with wave function $\psi_{310}(\vec{r})$ to the ground state. Calculate the wavelength of the emitted photon.
(You may recall that the energy of an electron with wave function $\psi_{nlm}(\vec{r})$ is proportional to $1/n^2$ and that the ionisation energy of an electron in the ground state ($n = 1$) is 13.6 eV. 1 eV=1.602×10^{-19} J.) [6]

END OF PAPER

