SEMESTER 1 EXAMINATION 2014-2015

WAVE PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. Explain what are meant by *travelling* and *standing* waves. [2]

Travelling waves are those which maintain a constant form that is simply translated through space as time evolves. Standing waves maintain a spatially fixed form, that is multiplied by an evolving function of time. Standing waves may be written as superpositions of travelling waves, and vice-versa.

Write an expression for a sinusoidal example of a travelling wave, and derive from it the *phase velocity*. [2]

A suitable sinusoidal example would be $\psi(x,t) = \psi_0 \sin(kx - \omega t + \varphi)$. *The phase velocity is then the velocity of a point of constant amplitude, i.e.* $\psi(x,t) = \psi_1$, *from which it follows that* $kx - \omega t + \varphi = c$, *which may be rearranged to give an equation of motion* $x = c/k + (\omega/k)t$. *The coefficient of* t *is the phase velocity.*

A2. Explain, with examples, the difference between *transverse* and *longitudinal* waves. [3]

Transverse and longitudinal waves refer to the propagation of vector quantities. A transverse wave is thus one in which the propagating quantity is a vector that is normal to the propagation direction [1], such as the displacement of a string or an electric/magnetic field component [0.5]. A longitudinal wave is similarly one in which the propagating quantity is a vector parallel to the propagation direction [1], such as the acoustic displacement of a medium or the direction of heat flow in a thermal wave [0.5].

Give an example of a wave that is neither transverse nor longitudinal. [1]

The quantum wavefunction; a chemical wave of species concentration; a 'wave of fear'.

A3. Outline the *Huygens description* of wave propagation. [2]

The Huygens description allows the propagation of a wavefront to be determined by placing imaginary sources along a given wavefront and calculating the disturbance that would result some time later from those sources alone. When performed geometrically, the new wavefront lies along the common tangent to the circular wavefronts from adjacent contributions.

Explain how the Huygens description can be used to calculate the diffraction pattern of an illuminated object. [2]

Imaginary sources are placed along a wavefront as it encounters the diffracting object, and the disturbance radiated by each source is considered to be modulated according to the transmission of the object at that point. The diffracted wave is then the resultant of the transmitted amplitudes.

A4. Explain how *dispersion* is apparent in the evolution of a propagating wavepacket, and in the phase velocities of its sinusoidal components. [2]

Dispersion describes the spreading of a wave packet as it propagates, and corresponds to a variation in the phase velocity as a function of the frequency of sinusoidal components.

The dispersion relation between the angular frequency ω and wavenumber k for the quantum wavefunction of a particle of mass m is

$$\omega = \frac{\hbar}{2m}k^2.$$

Determine the phase velocity and the group velocity for a wavepacket of (mean) wavenumber k. [2]

The phase velocity is given by

$$v_p = \frac{\omega}{k} = \frac{\hbar}{2m}k.$$

The group velocity is given by

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{\hbar}{m}k.$$

A5. Outline the *bandwidth theorem*, and explain its significance for both classical and quantum mechanical wave motions. [4]

The bandwidth theorem is that if we wish to limit the extent of a wavepacket in one dimension (space, time, frequency, wavenumber), then it will span at least a certain range in the space of the conjugate variable. A brief pulse therefore comprises a wide range of frequencies [1]. If we express the spread or range of the wavepacket in the space of a given variable by the uncertainty in the variable, then the bandwidth theorem is that product of the uncertainties cannot fall below a given value. For example [1],

$$\begin{aligned} \Delta_x \Delta_k &\geq \frac{1}{2}, \\ \Delta_t \Delta_\omega &\geq \frac{1}{2}. \end{aligned}$$

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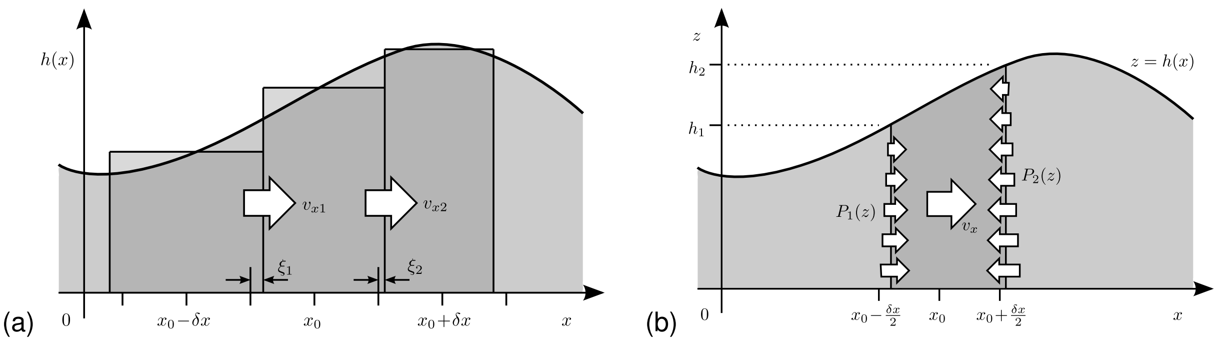
In classical systems, the bandwidth theorem hence means that to discriminate between similar frequencies there is a minimum duration for the measurement, and that for an instrument to respond to brief pulses it must have at least a certain bandwidth [1]. In quantum systems, the bandwidth theorem is equivalent to Heisenberg's uncertainty principle, that one cannot simultaneously know, for example, the position and momentum of an electron or photon, with arbitrary precision [1].

Section B

B1. Figure (a) shows how a wave in shallow water may be analysed by dividing the water into vertical slices of rest width δx and considering the motions of the slices. Here, x is the horizontal distance, $h(x)$ the water height, $\xi_{1,2}$ the displacements of the slice edges from their rest positions, and $v_{x1,2}$ the horizontal velocities of the edges. Motion is assumed limited to the $x-h$ plane.



(a) By assuming that the volume of water within each slice remains fixed, show that $h(x)\,(\delta x+\xi_2-\xi_1)$ will be constant, and hence that

$$\frac{\partial h}{\partial t} = -h_0\frac{\partial v_x}{\partial x},$$

where h_0 is the undisturbed height. Make clear any other assumptions. [4]

The distance between the edges of the slice will be $(\delta x+\xi_2-\xi_1)$. The volume of the slice will therefore be $h(x)(\delta x+\xi_2-\xi_1)\delta y$, where δy is the extent of the slice along the wavefront, which is assumed not to change as the wave propagates. For the quantity of water to remain constant, it hence follows that $h(x)(\delta x+\xi_2-\xi_1)$ will be constant. *[1]*

Differentiating with respect to time, we obtain

$$h(x)\left(\frac{\mathrm{d}\xi_2}{\mathrm{d}t}-\frac{\mathrm{d}\xi_1}{\mathrm{d}t}\right)+\frac{\partial h}{\partial t}(\delta x+\xi_2-\xi_1)=0.$$ *[1]*

Rearranging, assuming that $|\delta x|\gg|\xi_1-\xi_2|$, writing $v_{x1,x2}\equiv \mathrm{d}\xi_{1,2}/\mathrm{d}t$, taking the limit as $\delta x\to 0$ and assuming $h(x)\approx h_0$, we obtain

$$\begin{aligned}\frac{\partial h}{\partial t} &= -\lim_{\delta x\to 0} h(x)\frac{\frac{\mathrm{d}\xi_2}{\mathrm{d}t}-\frac{\mathrm{d}\xi_1}{\mathrm{d}t}}{\delta x}\\ &= -h(x)\lim_{\delta x\to 0}\frac{v_{x2}-v_{x1}}{\delta x}\\ &= -h(x)\frac{\partial v}{\partial x}\approx -h_0\frac{\partial v_x}{\partial x}.\end{aligned}$$

[2]

(b) Write an expression for the hydrostatic pressure $P_{1,2}(z)$ upon the edges of the slice, as shown in Figure (b) above. By considering the force upon a vertically thin element of the slice at height z, hence show that

$$\frac{\partial v_x}{\partial t} = -g\frac{\partial h}{\partial x},$$

where g is the acceleration due to gravity. [3]

The hydrostatic pressure at height z will be $(h(x) - z)\rho g$, where ρ is the water density and g the acceleration due to gravity. The water is assumed to be incompressible. The force upon a vertically thin element of thickness δz that extends a distance δy along the wavefront will therefore be

$$F = \{(h(x) - z) - (h(x + \delta x) - z)\}\,\rho g.$$ *[1]*

The mass of the element will be $\delta x \delta y \delta z \rho$. From Newton's second law, we hence obtain

$$\delta x \delta y \delta z \rho \frac{\partial^2 \xi}{\partial t^2} = \{(h(x) - z) - (h(x + \delta x) - z)\}\,\rho g.$$ *[1]*

Rearranging and taking the limit as $\delta x \to 0$, we hence obtain

$$\frac{\partial v_x}{\partial t} \equiv \frac{\partial^2 \xi}{\partial t^2} = -g \lim_{\delta x \to 0} \frac{h(x + \delta x) - h(x)}{\delta x} = -g\frac{\partial h}{\partial x}.$$ *[1]*

(c) Hence derive the wave equation for shallow-water waves

$$\frac{\partial^2 h}{\partial t^2} = gh_0\frac{\partial^2 h}{\partial x^2}.$$ [3]

Differentiating the two expressions with respect to t [1] and x [1], and equating the terms $\partial^2 v_x/\partial x \partial t$, we obtain *[2]*

$$\frac{\partial^2 h}{\partial t^2} = -h_0\frac{\partial^2 v_x}{\partial x \partial t} = -h_0\frac{\partial}{\partial x}\frac{\partial v_x}{\partial t} = gh_0\frac{\partial^2 h}{\partial x^2}.$$ *[1]*

(d) By substituting into the wave equation a trial travelling wave of the form $h(x,t) = h(u)$ where $u \equiv x - v_p t$, show that the phase velocity will be

$$v_p = \pm\sqrt{gh_0}.$$ [3]

Differentiation by the chain rule gives, for constant $\partial u/\partial t$ and $\partial u/\partial x$,

$$\frac{\partial^2 h(u)}{\partial t^2} = \left(\frac{\partial u}{\partial t}\right)^2 \frac{\mathrm{d}^2 h}{\mathrm{d}u^2} = (-v_p)^2 \frac{\mathrm{d}^2 h}{\mathrm{d}u^2}$$ *[1]*

and

$$gh_0 \frac{\partial^2 h}{\partial x^2} = gh_0 \left(\frac{\partial u}{\partial x}\right)^2 \frac{\mathrm{d}^2 h}{\mathrm{d}u^2} = gh_0 \frac{\mathrm{d}^2 h}{\mathrm{d}u^2}.$$ *[1]*

Substitution into the wave equation and cancellation of the common second derivative hence gives

$$v_p^2 = gh_0,$$

and hence the result sought. *[1]*

(e) Explain what happens as the straight wavefronts of ocean swell approach a gently shelving shoreline. [3]

As the wavefronts enter shallowing water, the reduction in h causes v_p to reduce [1], allowing the trailing parts of the wavefront to catch up with the more advanced parts [1]. The result is that, as they approach the shore, wavefronts become closer together and more parallel to the shore [1]. *[3]*

(f) Given that the energy density of shallow-water waves per unit horizontal (seabed) area is

$$\mathcal{E} = \rho g (h - h_0)^2,$$

determine the power per unit length along the wavefront. [2]

The power per unit wavefront length, $\mathcal{P}$, will be simply the product of the energy density and the wave velocity v_p,

$$\mathcal{P} = \rho g (h - h_0)^2 \sqrt{gh_0}.$$ *[2]*

(g) Hence explain how a wave originating in the deep ocean is transformed as it approaches the shore to become a *tsunami*. [2]

Unless energy is stored somewhere, the power into and out of any region must balance, so $\mathcal{P}$ must remain constant as the wave approaches the shore. As h_0 falls, there must therefore be an increase in the wave height $(h - h_0)$, which will grow in proportion to $h_0^{-1/4}$. *[2]*

B2. (a) Explain the principles of Fourier synthesis and analysis, and what is meant by the Fourier transform. [4]

The principle of Fourier synthesis is that any wavefunction can be built up from sinusoidal wave components of appropriate magnitudes and phases. The corresponding principle of Fourier analysis is that any wavefunction can be broken down into these components. If we know how sinusoidal wave components behave in a particular system, we can hence determine the behaviour of an arbitrary wave motion by breaking it into sinoisoidal components, allowing for their known behaviour, and recombining them into the composite wave. *[2]*

The Fourier transform is the mathematical operation that allows a function of time or position to be instead represented as a function of frequency or spatial frequency – i.e., by the spectrum of sinusoidal or complex exponential components into which it may be resolved. *[2]*

(b) A function $\psi(x)$ that is antisymmetrical about $x = 0$ and periodic with interval X may be written as

$$\psi(x) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{2\pi m}{X}x\right) \qquad (1)$$

where the Fourier components a_m are given by

$$a_m = \frac{2}{X}\int_{-X/2}^{X/2} \psi(x)\sin\left(\frac{2\pi m}{X}x\right)\mathrm{d}x.$$

Show that, for a square wave of interval X, defined for $|x| < X/2$ by

$$\begin{aligned}\psi(x) &= -a_0 \qquad (|x| < 0)\\ \psi(x) &= a_0 \qquad (|x| > 0),\end{aligned}$$

the Fourier components are given by

$$a_m = \frac{4a_0}{\pi m}\sin^2\left(\frac{m\pi}{2}\right).$$ [4]

We insert the given waveform into the expression for the Fourier components a_m, and break it into two parts at the discontinuity in $\psi(x)$:

$$\begin{aligned}a_m &= \frac{2}{X}\int_{-X/2}^{X/2} \psi(x)\sin\left(\frac{2\pi m}{X}x\right)\mathrm{d}x\\ &= \frac{2}{X}\left\{\int_{-X/2}^{0} -a_0\sin\left(\frac{2\pi m}{X}x\right)\mathrm{d}x + \int_{0}^{X/2} -a_0\sin\left(\frac{2\pi m}{X}x\right)\mathrm{d}x\right\}.\end{aligned}$$

The integrals are then straightforward to evaluate: *[2]*

$$
\begin{aligned}
a_m &= \frac{2a_0}{X}\frac{X}{2\pi m}\left\{\left[-\cos\left(\frac{2\pi m}{X}x\right)\right]_{-X/2}^{0} - \left[-\cos\left(\frac{2\pi m}{X}x\right)\right]_{0}^{X/2}\right\}. \\
&= \frac{a_0}{\pi m}\{-1 + (\cos\pi m) + \cos(\pi m) - 1\} \\
&= \frac{2a_0}{\pi m}(1 - \cos\pi m) = \frac{4a_0}{\pi m}\sin^2\left(\frac{m\pi}{2}\right).
\end{aligned}
$$

using $\cos 2\vartheta = 1 - 2\sin^2\vartheta$. *[2]*

(c) By integrating equation (1) over the range from $x = -X/4$ to x, show that a symmetrical triangular wave $\varphi(x')$ of period X, with a maximum at $x = 0$ and peak-to-peak amplitude $2b_0$ may be written as

$$\varphi(x') = \sum_{m=1}^{\infty} b_m \cos\left(\frac{2\pi m}{X}x'\right),$$

where

$$b_m = \frac{8b_0}{(\pi m)^2}\sin^2\left(\frac{m\pi}{2}\right).$$ [6]

Integrating equation (1) with respect to x *over the range from* $-X/4$ *to* x, *[2]*

$$
\begin{aligned}
\int_0^x \psi(x)\mathrm{d}x &= \sum_{m=1}^{\infty} a_m \int_{-X/4}^{x} \sin\left(\frac{2\pi m}{X}x\right)\mathrm{d}x \\
&= \sum_{m=1}^{\infty}\frac{X}{2\pi m}a_m\left[-\cos\left(\frac{2\pi m}{X}x\right)\right]_{-X/4}^{x} \\
&= \sum_{m=1}^{\infty}\frac{-X}{2\pi m}a_m\cos\left(\frac{2\pi m}{X}x\right),
\end{aligned}
$$

where we use that $\cos(\pi m/2) = 0$ *for odd* m *for which* $a_m \neq 0$.

The left side, since $\psi(x)$ *is a square wave, will be a triangular wave of the same periodicity, with maximum/minimum values of* $\pm a_0X/4$ *occurring (modulo* X*) at* $x = 0, X/2$. *To form the symmetrical triangular wave of peak-to-peak amplitude* $2b_0$, *we must therefore multiply by* $[-b_0/(a_0X/4)]$. *[2]*

We hence obtain

$$
\begin{aligned}
\varphi(x) &= \frac{-4b_0}{a_0X}\sum_{m=1}^{\infty}\frac{-X}{2\pi m}a_m\cos\left(\frac{2\pi m}{X}x\right) \\
&= \frac{2b_0}{\pi a_0}\sum_{m=1}^{\infty}\frac{4a_0}{\pi m^2}\sin^2\left(\frac{\pi m}{2}\right)\cos\left(\frac{2\pi m}{X}x\right) \\
&= \sum_{m=1}^{\infty}\frac{8b_0}{(\pi m)^2}\sin^2\left(\frac{\pi m}{2}\right)\cos\left(\frac{2\pi m}{X}x\right).
\end{aligned}
$$

[2]

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(d) The string of a musical instrument is plucked at its midpoint $x = 0$ in such a way that it is released from rest with a maximum displacement b_0 at time $t = 0$. The subsequent motion may be written as

$$\varphi(x,t) = \sum_{m=1}^{\infty} b_m \cos\left(\frac{2\pi m}{X}x\right)\cos\left(\frac{2\pi m}{T}t\right),$$

where T is the period of the fundamental oscillation, the coefficients b_m are as defined in part (c), and the phase velocity for waves on the string is given by $v_p = X/T$.

Show that, at an arbitrary point x, the velocity of the string $\partial\varphi/\partial t$ may be written as a superposition of two square waveforms of period T and amplitude $\pm 2b_0/T$ with a relative delay of $(2x/X)T$. [4]

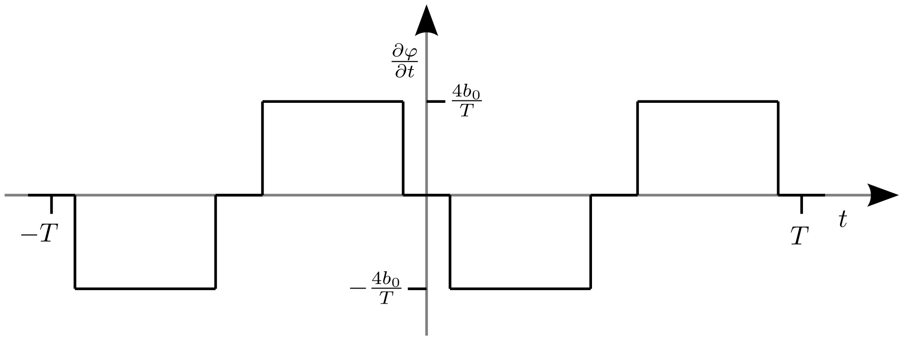
Differentiating the expression for $\varphi(x,t)$ and writing b_m in terms of a_m, the string velocity will be

$$\begin{aligned}
\frac{\partial\varphi}{\partial t} &= \sum_{m=1}^{\infty} b_m \cos\left(\frac{2\pi m}{X}x\right)\left(-\frac{2\pi m}{T}\right)\sin\left(\frac{2\pi m}{T}t\right) \\
&= \sum_{m=1}^{\infty} \frac{2b_0}{\pi m a_0} a_m \left(-\frac{2\pi m}{T}\right)\cos\left(\frac{2\pi m}{X}x\right)\sin\left(\frac{2\pi m}{T}t\right) \\
&= \left(-\frac{4b_0}{a_0 T}\right)\sum_{m=1}^{\infty} a_m \sin\left(\frac{2\pi m}{T}t\right)\cos\left(\frac{2\pi m}{X}x\right) \\
&= \left(-\frac{4b_0}{a_0 T}\right)\sum_{m=1}^{\infty} \frac{a_m}{2}\left\{\sin\left(\frac{2\pi m}{T}t + \frac{2\pi m}{X}x\right) + \sin\left(\frac{2\pi m}{T}t - \frac{2\pi m}{X}x\right)\right\} \\
&= \left(-\frac{2b_0}{a_0 T}\right)\left\{\sum_{m=1}^{\infty} a_m \sin\left[\frac{2\pi m}{T}\left(t + \frac{Tx}{X}\right)\right] + \sum_{m=1}^{\infty} a_m \sin\left[\frac{2\pi m}{T}\left(t - \frac{Tx}{X}\right)\right]\right\}.
\end{aligned}$$

[Marks for lines 1, 3, 5] *[3]*

Each summation is an (inverted) version of the square wave of equation (1) that alternates between $\pm 2b_0/T$, and is advanced or delayed by Tx/X to give a relative delay of $(2x/X)T$. *[1]*

(e) Hence sketch the velocity of the string at point $x = X/16$ for $-T \le t \le T$. [2]



[2]

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B3. (a) Explain what is meant by the *impedance* of a medium in the context of wave propagation. [2]

The impedance is a measure of the resistance of the medium to disturbance by the process driving the wave motion. It is related to the ratio of the two properties that are conserved at an interface, and therefore determines the reflectivity at such a boundary: if the impedances are the same on both sides of the interface, the wave is not reflected. *[2]*

(b) The continuity conditions for electromagnetic waves normally incident upon the plane interface between two media are

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{E}_2 \\ \mathbf{H}_1 &= \mathbf{H}_2, \end{aligned}$$

where $\mathbf{E}_{1,2}$ and $\mathbf{H}_{1,2}$ are the total electric and magnetic field strengths in the two media at the interface and, for a wave component travelling in direction $\hat{\mathbf{n}}$, the magnetic field strength $\mathbf{H} = (1/Z)\,\mathbf{E}\times\hat{\mathbf{n}}$, where Z is the impedance of the medium.

By considering wave components that are incident upon, reflected by and transmitted through the interface, derive the amplitude reflection coefficient for electromagnetic waves in terms of the impedances Z_1 and Z_2. [3]

We write incident, reflected and transmitted wave components at the interface at time t as the electric field strengths $\mathbf{E}_i(t)$, $\mathbf{E}_r(t)$ and $\mathbf{E}_t(t)$, and hence the magnetic field strengths $\mathbf{H}_i(t) = Z_1^{-1}\mathbf{E}_i(t)\times\hat{\mathbf{n}}$, $\mathbf{E}_r(t) = Z_1^{-1}\mathbf{E}_r(t)\times\hat{\mathbf{n}}$ and $\mathbf{E}_t(t) = Z_2^{-1}\mathbf{E}_t(t)\times\hat{\mathbf{n}}$. Applying the continuity conditions for normal incidence, we therefore obtain, for components in the direction of the transverse electric and magnetic fields

$$\begin{aligned} E_i + E_r &= E_t \\ \frac{1}{Z_1}(E_i - E_r) &= \frac{1}{Z_2}E_t. \end{aligned}$$

[1]

Combining these to eliminate E_t, we obtain

$$Z_2(E_i - E_r) = Z_1(E_i + E_r)$$ *[1]*

and hence, rearranging for E_r, we obtain the amplitude reflection coefficient,

$$\frac{E_r}{E_i} = \frac{Z_2 - Z_1}{Z_1 + Z_2}.$$ *[1]*

(c) Deduce further expressions for the ratio $|\mathbf{E}_t/\mathbf{E}_i|$ of the transmitted and incident electric fields $\mathbf{E}_{t,i}$, and for the ratio $|\mathbf{H}_t/\mathbf{H}_i|$ of the transmitted and incident magnetic fields $\mathbf{H}_{t,i}$. [2]

Using our result for E_r/E_i, or re-solving the simultaneous equations, we obtain

$$E_t/E_i = \frac{2Z_2}{Z_1 + Z_2}$$

and hence

$$H_t/H_i = \frac{2Z_1}{Z_1 + Z_2}.$$

[2]

(d) Show that, if $Z_2 \gg Z_1$, the ratio $|\mathbf{H}_t/\mathbf{H}_i| \approx 0$, and hence that the incident and reflected magnetic field components must be equal and opposite. Show that, conversely, if $Z_2 \ll Z_1$, the electric field components must cancel. [3]

$H_t/H_i = (2Z_1/Z_2)/(1 + Z_1/Z_2)$. Hence if $Z_2 \gg Z_1$, it follows that $H_t/H_i \approx 2 \times 0/(1 + 0) = 0$. From the second continuity condition that $\mathbf{H}_1 = \mathbf{H}_2$, since $\mathbf{H}_2 \approx 0$, it follows that $\mathbf{H}_i + \mathbf{H}_r \approx 0$ and hence $\mathbf{H}_i \approx -\mathbf{H}_r$; the incident and reflected magnetic field components must hence be approximately equal and opposite. *[2]*

Conversely, if $Z_2 \ll Z_1$, we find that $E_t/E_i \approx 0$ and hence, from the first continuity condition, $\mathbf{E}_i \approx -\mathbf{E}_r$, so it is the electric field components that must cancel. *[1]*

(e) Newton observed his 'rings' by placing a lens of refractive index $\eta = 1.55$ onto a block of the same material so that its lower surface of radius of curvature $R = 2.3\,\mathrm{m}$ touched the plane surface of the glass block. When the lens was illuminated from above with yellow-orange light, and viewed from the same direction, a concentric series of finely spaced bright and dark rings was observed.

Explain the origin of the observed ring pattern, and the reason why the centre of the fringe pattern was dark rather than bright. [5]

TURN OVER

Illumination from above is reflected back to the viewer by both the glass-air interface at the lower face of the lens, and the air-glass interface at the top of the block. The fringes result from the interference of these two waves. According to the path difference - which is twice the thickness of the air gap at the observed position - the two waves will either add constructively, resulting in a bright fringe, or destructively, leaving darkness. *[2]*

At the centre of the pattern where the lens and block touch and hence the thickness of the air gap is zero, the two waves would be in phase but for a difference between the phases introduced at the reflecting interfaces [1]. For the glass-air interface, and passage into a medium of higher impedance (for electromagnetic waves, $Z = \sqrt{\mu/\varepsilon}$), the incident and reflected electric fields have the same sign, while for the air-glass interface the signs are opposites [1]. The difference in sign is equivalent to an additional phase difference of π, and hence the centre of the pattern shows destructive interference between the reflected components [1]. (In the absence of an air gap the medium is effectively continuous, and there is therefore nothing to cause a reflection.) *[3]*

(f) Show that, if the wavelength of illumination is λ, the radius r_n of the nth dark fringe will be approximately given by

$$r_n \approx \sqrt{nR\lambda}.$$ [3]

You may neglect the effects of refraction throughout.

(A picture will help here...)

The thickness of the air gap at radius r will be

$$t = R - \sqrt{R^2 - r^2} = R\left(1 - (1 - (r/R)^2\right)^{1/2} \approx R\left((1/2)(r/R)^2\right).$$ *[2]*

For a dark fringe, taking into account the phase difference discussed above, we require $2t = n\lambda$, from which it follows that

$$2\frac{R}{2}\left(\frac{r_n}{R}\right)^2 \approx n\lambda$$

and hence $r_n \approx \sqrt{nR\lambda}$. *[1]*

(g) Newton measured the radius of the fifth dark ring to be 2.57 mm. Deduce the wavelength of the orange-yellow light. [2]

Rearranging for λ and substituting the values given, we obtain

$$\lambda = \frac{r_n^2}{nR} = \frac{0.00257^2}{5 \times 2.3}\,\mathrm{m} = 574\,\mathrm{nm}.$$ *[2]*

B4. A source of waves of angular frequency ω_s moves with a velocity $\mathbf{v}$ and, at time $t = 0$, is at a position $\mathbf{r}_0$ relative to a stationary observer.

(a) Show that the distance from the source to the observer at time $t \ll |\mathbf{r}_0|/|\mathbf{v}|$ will be given approximately by

$$r \approx |\mathbf{r}_0| + \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \mathbf{v}t.$$ [2]

If $t \ll |\mathbf{r}_0|/|\mathbf{v}|$, the change in the bearing of the source from the observer will be negligible, and hence the distance will change only by the component of the change of position $\mathbf{v}t$ along the unit position vector $\mathbf{r}_0/|\mathbf{r}_0|$. Adding this correction to the initial value, we obtain the expression given. *[2]*

(b) Show therefore that if the wave leaving the source at time t is $\psi(t)$, then that seen by the observer will be proportional to

$$\psi\left(t - t_0 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c}t\right),$$

where $t_0 = |\mathbf{r}_0|/c$ and c is the speed with which the wave propagates. [3]

We assume that the observed wave will be proportional to $\psi(t - \tau)$ [1], where $\tau = r/c$ is the time it takes the wave to travel from the source to the observer at speed c [1]. It follows that the observed wave will be proportional to

$$\psi\left(t - \frac{|\mathbf{r}_0|}{c} - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}t}{c}\right)$$

as required [1]. *[3]*

(c) Hence show that the observed wave will have an angular frequency $\omega_s - \delta\omega$, where

$$\frac{\delta\omega}{\omega_s} = \frac{v_x}{c},$$

and v_x is the component of the source's velocity away from the observer. [3]

If $\psi(t) = \psi_0 \cos \omega_s t$, then the observed wave will be proportional to

$$\psi_0 \cos\left(\omega_s\left[t - t_0 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c}t\right]\right) = \psi_0 \cos\left(\omega_s\left[1 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c}\right]t - \omega_s t_0\right).$$ *[1]*

TURN OVER

The observed frequency s the coefficient of t in this expression, and hence

$$\omega_s - \delta\omega = \omega_s \left[1 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c}\right] \qquad [\,1\,]$$

where $\mathbf{v} \cdot \mathbf{r}_0/|\mathbf{r}_0| = v_x$ is the velocity component away from the observer, hence

$$\delta\omega/\omega_s = v_x/c. \qquad [\,1\,]$$

The source is an atom which, when at rest, emits or scatters photons of angular frequency ω_0. The atom emits a photon towards the observer, in whose frame it has an energy $\hbar\omega$. The coordinate axes may be chosen so that the x axis points from the source to the observer.

(d) By considering the total electronic and kinetic energy of the atom before and after the emission of the photon, show that, if the x-component of the atom's velocity changes by δv when it emits the photon, conservation of energy requires that

$$\hbar\omega = \hbar\omega_0 - mv_x\delta v,$$

where m is the mass of the atom and v_x the mean component of its velocity away from the observer. [4]

If the initial and final velocity components are $v_x - \delta v/2$ and $v_x + \delta v/2$, then the initial and final energies of the atom will be $\hbar\omega_0 + m(v_x - \delta v/2)^2/2$ and $m(v_x + \delta v/2)^2/2$. [2]

The change in atomic energy, which will be emitted as the photon of frequency ω, will hence be

$$\begin{aligned} \hbar\omega &= \hbar\omega_0 + \frac{m}{2}\left(v_x - \frac{\delta v}{2}\right)^2 - \frac{m}{2}\left(v_x + \frac{\delta v}{2}\right)^2 \\ &= \hbar\omega_0 + mv_x\delta v. \end{aligned}$$

[2]

(e) Show that, if momentum is conserved during the emission of the photon,

$$m\delta v = \hbar\omega/c. \qquad [\,2\,]$$

The initial and final momenta of the atom will be $m(v_x - \delta v/2)$ and $m(v_x + \delta v/2)$. The change in momentum, which will be transferred to the backward-emitted photon, will hence be

$$\hbar k = \hbar\frac{\omega}{c} = m\left(v_x + \frac{\delta v}{2}\right) - m\left(v_x - \frac{\delta v}{2}\right) = m\delta v. \qquad [\,2\,]$$

(f) Hence show that the observed angular frequency will be

$$\omega = \omega_0 \left(1 + \frac{v_x}{c}\right)^{-1},$$

and therefore that, if $v_x \ll c$, the Doppler shift of the photon due to the motion of the atom will again be

$$\delta\omega = \omega - \omega_0 \approx \omega_0 \frac{v_x}{c}.$$ [3]

Substituting the result from (e) into the expression from (d), we find

$$\hbar\omega = \hbar\omega_0 + m v_x \hbar \frac{\omega}{c} \frac{1}{m}$$

so, collecting together the terms in ω,

$$\hbar\omega \left(1 - \frac{c_x}{c}\right) = \hbar\omega_0,$$ *[1]*

hence the result required. If $v_x \ll c$, we may use the binomial expansion to obtain

$$\omega \approx \omega_0 \left(1 - \frac{v_x}{c}\right)$$ *[1]*

and hence

$$\delta\omega \equiv \omega - \omega_0 \approx \omega_0 \left(1 - \frac{v_x}{c}\right) - \omega_0 = \omega_0 \frac{v_x}{c}.$$ *[1]*

The Fraunhofer K line in the solar spectrum is due to absorption at wavelength $\lambda_0 = 394\,\mathrm{nm}$ by Ca^+ ions in the photosphere, where the temperature T is around 5 000 K.

(g) Estimate the variation $\delta\lambda$ in the wavelength of the K line that is due to thermal motion of the Ca^+ ions. The r.m.s. velocity component $v_{x,rms}$ for a thermal distribution is given by $v_{x,rms}^2 = k_B T/m$, where k_B is Boltzmann's constant and the mass m of a Ca^+ ion is 6.66×10^{-26} kg. [3]

You may assume that $\delta\lambda/\lambda_0 = \delta\omega/\omega_0$.

Using the data given,

$$v_{x,rms} = \sqrt{\frac{k_B T}{m}} = \sqrt{\frac{1.38\times10^{-23}\times5000}{6.66\times10^{-26}}}\,\mathrm{m\,s^{-1}} = 1020\,\mathrm{m\,s^{-1}}.$$ *[2]*

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Hence

$$\delta\lambda = \lambda_0 \frac{\delta\omega}{\omega_0} = 394 \times \frac{1020}{3 \times 10^8}\,\text{nm} = 0.0013\,\text{nm}.$$

[1]

END OF PAPER