SEMESTER 1 EXAMINATION 2014-2015

WAVE PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. Explain what are meant by *travelling* and *standing* waves. [2]

Write an expression for a sinusoidal example of a travelling wave, and derive from it the *phase velocity*. [2]

A2. Explain, with examples, the difference between *transverse* and *longitudinal* waves. [3]

Give an example of a wave that is neither transverse nor longitudinal. [1]

A3. Outline the *Huygens description* of wave propagation. [2]

Explain how the Huygens description can be used to calculate the diffraction pattern of an illuminated object. [2]

A4. Explain how *dispersion* is apparent in the evolution of a propagating wavepacket, and in the phase velocities of its sinusoidal components. [2]

The dispersion relation between the angular frequency ω and wavenumber k for the quantum wavefunction of a particle of mass m is

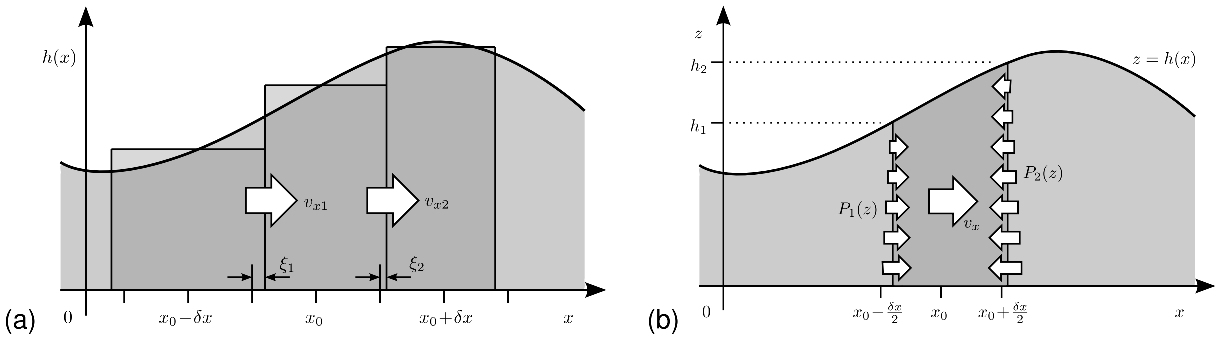
$$\omega = \frac{\hbar}{2m}k^2.$$

Determine the phase velocity and the group velocity for a wavepacket of (mean) wavenumber k. [2]

A5. Outline the *bandwidth theorem*, and explain its significance for both classical and quantum mechanical wave motions. [4]

Section B

B1. Figure (a) shows how a wave in shallow water may be analysed by dividing the water into vertical slices of rest width δx and considering the motions of the slices. Here, x is the horizontal distance, $h(x)$ the water height, $\xi_{1,2}$ the displacements of the slice edges from their rest positions, and $v_{x1,2}$ the horizontal velocities of the edges. Motion is assumed limited to the $x-h$ plane.



(a) By assuming that the volume of water within each slice remains fixed, show that $h(x)\,(\delta x + \xi_2 - \xi_1)$ will be constant, and hence that

$$\frac{\partial h}{\partial t} = -h_0 \frac{\partial v_x}{\partial x},$$

where h_0 is the undisturbed height. Make clear any other assumptions. [4]

(b) Write an expression for the hydrostatic pressure $P_{1,2}(z)$ upon the edges of the slice, as shown in Figure (b) above. By considering the force upon a vertically thin element of the slice at height z, hence show that

$$\frac{\partial v_x}{\partial t} = -g \frac{\partial h}{\partial x},$$

where g is the acceleration due to gravity. [3]

(c) Hence derive the wave equation for shallow-water waves

$$\frac{\partial^2 h}{\partial t^2} = g h_0 \frac{\partial^2 h}{\partial x^2}.$$ [3]

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(d) By substituting into the wave equation a trial travelling wave of the form $h(x, t) = h(u)$ where $u \equiv x - v_p t$, show that the phase velocity will be

$$v_p = \pm \sqrt{g h_0}.$$

[3]

(e) Explain what happens as the straight wavefronts of ocean swell approach a gently shelving shoreline. [3]

(f) Given that the energy density of shallow-water waves per unit horizontal (seabed) area is

$$\mathcal{E} = \rho g (h - h_0)^2,$$

determine the power per unit length along the wavefront. [2]

(g) Hence explain how a wave originating in the deep ocean is transformed as it approaches the shore to become a *tsunami*. [2]

B2. (a) Explain the principles of Fourier synthesis and analysis, and what is meant by the Fourier transform. [4]

(b) A function $\psi(x)$ that is antisymmetrical about $x = 0$ and periodic with interval X may be written as

$$\psi(x) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{2\pi m}{X}x\right) \tag{1}$$

where the Fourier components a_m are given by

$$a_m = \frac{2}{X}\int_{-X/2}^{X/2} \psi(x)\sin\left(\frac{2\pi m}{X}x\right)\mathrm{d}x.$$

Show that, for a square wave of interval X, defined for $|x| < X/2$ by

$$\begin{aligned} \psi(x) &= -a_0 \qquad (|x| < 0) \\ \psi(x) &= a_0 \qquad (|x| > 0), \end{aligned}$$

the Fourier components are given by

$$a_m = \frac{4a_0}{\pi m}\sin^2\left(\frac{m\pi}{2}\right).$$ [4]

(c) By integrating equation (1) over the range from $x = -X/4$ to x, show that a symmetrical triangular wave $\varphi(x')$ of period X, with a maximum at $x = 0$ and peak-to-peak amplitude $2b_0$ may be written as

$$\varphi(x') = \sum_{m=1}^{\infty} b_m \cos\left(\frac{2\pi m}{X}x'\right),$$

where

$$b_m = \frac{8b_0}{(\pi m)^2}\sin^2\left(\frac{m\pi}{2}\right).$$ [6]

(d) The string of a musical instrument is plucked at its midpoint $x = 0$ in such a way that it is released from rest with a maximum displacement b_0 at time $t = 0$. The subsequent motion may be written as

$$\varphi(x,t) = \sum_{m=1}^{\infty} b_m \cos\left(\frac{2\pi m}{X}x\right)\cos\left(\frac{2\pi m}{T}t\right),$$

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where T is the period of the fundamental oscillation, the coefficients b_m are as defined in part (c), and the phase velocity for waves on the string is given by $v_p = X/T$.

Show that, at an arbitrary point x, the velocity of the string $\partial\varphi/\partial t$ may be written as a superposition of two square waveforms of period T and amplitude $\pm 2b_0/T$ with a relative delay of $(2x/X)T$. [4]

(e) Hence sketch the velocity of the string at point $x = X/16$ for $-T \leq t \leq T$. [2]

B3. (a) Explain what is meant by the *impedance* of a medium in the context of wave propagation. [2]

(b) The continuity conditions for electromagnetic waves normally incident upon the plane interface between two media are

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{E}_2 \\ \mathbf{H}_1 &= \mathbf{H}_2, \end{aligned}$$

where $\mathbf{E}_{1,2}$ and $\mathbf{H}_{1,2}$ are the total electric and magnetic field strengths in the two media at the interface and, for a wave component travelling in direction $\hat{\mathbf{n}}$, the magnetic field strength $\mathbf{H} = (1/Z)\,\mathbf{E} \times \hat{\mathbf{n}}$, where Z is the impedance of the medium.

By considering wave components that are incident upon, reflected by and transmitted through the interface, derive the amplitude reflection coefficient for electromagnetic waves in terms of the impedances Z_1 and Z_2. [3]

(c) Deduce further expressions for the ratio $|\mathbf{E}_t/\mathbf{E}_i|$ of the transmitted and incident electric fields $\mathbf{E}_{t,i}$, and for the ratio $|\mathbf{H}_t/\mathbf{H}_i|$ of the transmitted and incident magnetic fields $\mathbf{H}_{t,i}$. [2]

(d) Show that, if $Z_2 \gg Z_1$, the ratio $|\mathbf{H}_t/\mathbf{H}_i| \approx 0$, and hence that the incident and reflected magnetic field components must be equal and opposite. Show that, conversely, if $Z_2 \ll Z_1$, the electric field components must cancel. [3]

(e) Newton observed his 'rings' by placing a lens of refractive index $\eta = 1.55$ onto a block of the same material so that its lower surface of radius of curvature $R = 2.3\,\mathrm{m}$ touched the plane surface of the glass block. When the lens was illuminated from above with yellow-orange light, and viewed from the same direction, a concentric series of finely spaced bright and dark rings was observed.

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Explain the origin of the observed ring pattern, and the reason why the centre of the fringe pattern was dark rather than bright. [5]

(f) Show that, if the wavelength of illumination is λ, the radius r_n of the nth dark fringe will be approximately given by

$$r_n \approx \sqrt{nR\lambda}.$$ [3]

You may neglect the effects of refraction throughout.

(g) Newton measured the radius of the fifth dark ring to be 2.57 mm. Deduce the wavelength of the orange-yellow light. [2]

B4. A source of waves of angular frequency ω_s moves with a velocity $\mathbf{v}$ and, at time $t = 0$, is at a position $\mathbf{r}_0$ relative to a stationary observer.

(a) Show that the distance from the source to the observer at time $t \ll |\mathbf{r}_0|/|\mathbf{v}|$ will be given approximately by

$$r \approx |\mathbf{r}_0| + \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \mathbf{v}t.$$

[2]

(b) Show therefore that if the wave leaving the source at time t is $\psi(t)$, then that seen by the observer will be proportional to

$$\psi\left(t - t_0 - \frac{\mathbf{r}_0}{|\mathbf{r}_0|} \cdot \frac{\mathbf{v}}{c}t\right),$$

where $t_0 = |\mathbf{r}_0|/c$ and c is the speed with which the wave propagates. [3]

(c) Hence show that the observed wave will have an angular frequency $\omega_s - \delta\omega$, where

$$\frac{\delta\omega}{\omega_s} = \frac{v_x}{c},$$

and v_x is the component of the source's velocity away from the observer. [3]

The source is an atom which, when at rest, emits or scatters photons of angular frequency ω_0. The atom emits a photon towards the observer, in whose frame it has an energy $\hbar\omega$. The coordinate axes may be chosen so that the x axis points from the source to the observer.

(d) By considering the total electronic and kinetic energy of the atom before and after the emission of the photon, show that, if the x-component of the atom's velocity changes by δv when it emits the photon, conservation of energy requires that

$$\hbar\omega = \hbar\omega_0 - mv_x\delta v,$$

where m is the mass of the atom and v_x the mean component of its velocity away from the observer. [4]

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(e) Show that, if momentum is conserved during the emission of the photon,

$$m\delta v = \hbar\omega/c. \qquad [2]$$

(f) Hence show that the observed angular frequency will be

$$\omega = \omega_0 \left(1 + \frac{v_x}{c}\right)^{-1},$$

and therefore that, if $v_x \ll c$, the Doppler shift of the photon due to the motion of the atom will again be

$$\delta\omega = \omega - \omega_0 \approx \omega_0 \frac{v_x}{c}. \qquad [3]$$

The Fraunhofer K line in the solar spectrum is due to absorption at wavelength $\lambda_0 = 394$ nm by Ca^+ ions in the photosphere, where the temperature T is around 5 000 K.

(g) Estimate the variation $\delta\lambda$ in the wavelength of the K line that is due to thermal motion of the Ca^+ ions. The r.m.s. velocity component $v_{x,rms}$ for a thermal distribution is given by $v^2_{x,rms} = k_B T/m$, where k_B is Boltzmann's constant and the mass m of a Ca^+ ion is 6.66×10^{-26} kg. [3]

You may assume that $\delta\lambda/\lambda_0 = \delta\omega/\omega_0$.

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