Section A

Bookwork

A1) T, U, S, and p are all functions of state [1/2] mark each

T and p are intensive variables

U and S are extensive variables

[1/2] mark each

Total [4] marks

A2) The statistical weight of each macrostate is $W = \frac{N!}{(n_\uparrow)!(n_\downarrow)!}$ [1/2] mark

similar to problem sheet

Statistical weight for $n_\uparrow = n_\downarrow = \frac{N}{2}$ is $W = \frac{N!}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}\right)!}$

[1/2] mark

$\sigma = \ln W$ [1/2] mark

Using Stirling's formula

$$\sigma = N\ln N - N - 2\left(\frac{N}{2}\ln\left(\frac{N}{2}\right) - \frac{N}{2}\right) + \frac{1}{2}\ln(2\pi N) - \ln(\pi N)$$

[1/2] mark

The last two terms can be combined

$$\Rightarrow \tfrac{1}{2}\ln 2 + \tfrac{1}{2}\ln(\pi N) - \ln(\pi N) = \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\ln(\pi N)$$ [1/2] mark

$$= \frac{1}{2}\ln\left(\frac{2}{\pi N}\right) = -\frac{1}{2}\ln\left(\frac{\pi N}{2}\right)$$

[1/2] mark

Hence $\sigma = N\ln N - N\ln\left(\frac{N}{2}\right) - \frac{1}{2}\ln\left(\frac{\pi N}{2}\right)$ [1/2] mark

$$\Rightarrow \sigma = N\ln 2 - \tfrac{1}{2}\ln\left(\frac{\pi N}{2}\right)$$ [1/2] mark

The entropy of all the macrostates combined (the total number of microstates) is 2^N

and so $\sigma_{tot} = \ln 2^N = N\ln 2$ [½] mark

since $\frac{1}{2}\ln\left(\frac{\pi N}{2}\right)$ is negligible compared to $N\ln 2$

then $\sigma \simeq \sigma_{tot}$ [½] mark

Total [5] marks

A3) Bookwork

In the expression for σ the only term that depends on E is $N\ln(E)^{3/2} = \frac{3}{2}N\ln E$ [½] mark

$$\frac{\partial\left(\frac{3N}{2}\ln E\right)}{\partial E} = \frac{3N}{2E}$$ [½] mark

Hence $\frac{1}{k_BT} = \frac{3N}{2E}$ and so $E = \frac{3}{2}Nk_BT$ [½] mark

As $T \to 0$, $E \to 0$ [½] mark and so $\ln E$ term in Sackur-Tetrode formula $\to -\infty$ as $T \to 0$.

[½] mark

This is in disagreement with the third law of thermodynamics which states that

$S = k_B\sigma \to$ constant (finite) as $T \to 0$

[½] mark

Total [3] marks

A4) Bookwork

$$Z = \sum_{n=0}^{\infty} e^{-\beta \epsilon_n} = e^{-\frac{\beta\hbar\omega}{2}} \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} = \frac{e^{-\frac{\beta\hbar\omega}{2}}}{1-e^{-\beta\hbar\omega}}$$

[1] [1] [1]

since $r = \exp(-\beta\hbar\omega)$ in $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ [1]

Total [4] marks

Bookwork

A5) At $T=0$ all energy levels are occupied for energy $\leq \epsilon_F$, [1] mark while all levels with energy $> \epsilon_F$ are not occupied. (Alternatively, ϵ_F is the energy of the most energetic fermion at $T=0$) [1] mark

At $T=0$

$$N = \int_{\epsilon \leq \epsilon_F} \frac{g_s V}{h^3} d^3p \longrightarrow \int_0^{p_F} \frac{g_s V p^2 d\Omega\, dp}{h^3}$$

(given in question) (change to Spherical coordinates in momentum space)

[1/2] for correct integration limits, and for correct integrand

Then integrate over $d\Omega$ over one octant (since $p_x, p_y, p_z > 0$)

$$N \longrightarrow \frac{4\pi}{8}\int_0^{p_F} \frac{8 g_s V p^2 dp}{h^3} = 4\pi \int_0^{p_F} \frac{g_s V p^2 dp}{h^3}$$ [1/2] mark

Integration over dp gives $n = \frac{g_s p_F^3}{2\pi^2 \hbar^3 \cdot 3} = \frac{g_s p_F^3}{6\pi^2 \hbar^3}$

[1/2] mark

rearranging gives $p_F = \hbar\left(\frac{6\pi^2 n}{g_s}\right)^{1/3}$

and $\epsilon_F = \frac{p_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g_s}\right)^{2/3}$ [1/2] mark

Total [4] marks

Total for section A

[4] + [5] + [3] + [4] + [4]

= [20]

marks

B1) All bookwork

a) $\Phi_G = U - TS - \mu N$ (given in question)

and so

$$d\Phi_G = dU - TdS - SdT - \mu dN - Nd\mu$$ (1) mark

Since $dU = TdS - pdV + \mu dN$ (1) mark

then $d\Phi_G = -SdT - pdV - Nd\mu$ (1) mark

Therefore $\Phi_G = \Phi_G(T, V, \mu)$

and one has

$$d\Phi_G = \left(\frac{\partial \Phi_G}{\partial T}\right)_{V,\mu} dT + \left(\frac{\partial \Phi_G}{\partial V}\right)_{T,\mu} dV + \left(\frac{\partial \Phi_G}{\partial \mu}\right)_{T,V} d\mu$$ (1) mark

Comparing coefficients of $d\mu$ in both expressions for $d\Phi_G$ one has $N = -\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{T,V}$ (1) mark

Total [5] marks

b) $\langle N \rangle = \sum_i N_i P_i = \sum_i \frac{N_i e^{\beta(\mu N_i - E_i)}}{Z}$ (1) mark

(hint given in question)

Therefore $\langle N \rangle = \frac{1}{\beta Z}\frac{\partial}{\partial \mu}\sum_i e^{\beta(\mu N_i - E_i)} = \frac{1}{\beta Z}\frac{\partial Z}{\partial \mu}$

(1) mark (1) mark

$\left(\frac{1}{\beta} = k_B T\right)$

$= k_B T \frac{\partial \ln Z}{\partial \mu}$ since $\frac{\partial \ln Z}{\partial \mu} = \frac{\partial Z}{\partial \mu}\frac{\partial \ln Z}{\partial Z}$

(1) mark $= \frac{1}{Z}\frac{\partial Z}{\partial \mu}$

(1) mark

Now $\underset{\substack{\downarrow \\ \text{statistical} \\ \text{mechanics}}}{\langle N \rangle} = \underset{\substack{\downarrow \\ \text{Thermodynamics}}}{N}$ (1) mark

Since $N = -\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{T,V}$ shown in part a)

then $-\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{T,V} = k_B T \frac{\partial \ln Z}{\partial \mu}$

and so $\Phi_G = -k_B T \ln Z$ (1) mark

Total [7] marks

c) $\Phi_G = -\frac{k_B T e^{\beta \mu} V}{\lambda_{th}^3}$ (1) mark

$N = -\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{T,V} = \frac{e^{\beta \mu} V}{\lambda_{th}^3}$ (1) mark rearranging gives $\mu = k_B T \ln(n\lambda_{th}^3)$ [1] mark

Total [3] marks

d) $\ln Z = \dfrac{e^{\beta\mu} V}{\lambda_{th}^3}$ (used in c) above)

$\dfrac{1}{\lambda_{th}^3} \sim \beta^{-3/2}$ [1] mark

$U = -\left(-\dfrac{3}{2}\, \dfrac{e^{\beta\mu} V}{\lambda_{th}^3 \beta}\right)$ [1] mark because $\beta\mu$ is a constant under differentiation [1] mark

and $\beta^{-3/2} \xrightarrow{\frac{\partial}{\partial\beta}} -\dfrac{3}{2}\beta^{-5/2}$ [1] mark

since $N = \dfrac{e^{\beta\mu} V}{\lambda_{th}^3}$ [1/2] mark (shown in c)

then $U = \frac{3}{2} N k_B T$ [1/2] mark

Total [5] marks

Total for B2

[5] + [7] + [3] + [5] = [20] marks

B.2) a) and b) bookwork c) from unassessed problem sheet

a) The thermal de Broglie wavelength can be interpreted as the average de Broglie wavelength [1] of the gas particles. Each particle has a de Broglie wavelength $\lambda = \frac{h}{p}$ where p is momentum [1] (waveparticle duality).

Total [2] marks

b) There is no limit on the left-hand side of the equation $n\lambda_{th}^3 = Li_{3/2}(z)$ [1] because both n [½] and T [½] can be chosen freely.

The right-hand side of the equation is bounded by 2.6 as $z \to 1$, and so the equation cannot be solved ~~as~~ for z as $z \to 1$ [1]

The reason why the equation cannot be solved for $z \to 1$ is because the approximation of going from a discrete sum over the occupation numbers of the single particle quantum states k_ε to an integral over the continuous variable x [1] is not counting the ground state, because the integrand in $Li_{3/2}(z)$ contains $x^{1/2}$, which $\to 0$ as $\epsilon \to 0$ [1] Total [5] marks

c) $\underset{\text{Total particle number}}{N} = \sum_k \underset{\text{average occupation number}}{\langle n_k \rangle} = \underset{\text{number in ground state}}{N_0} + \underset{\text{number in all other states}}{N_1}$

N_0: not accounted for in integral

N_1: correctly described by integral [1]

One has $n_1 \lambda_{th}^3 = Li_{3/2}(z)$ for $T \le T_c$ (or as $z \to 1$) [1]

so $$\frac{n_1 \lambda_{th}^3(T)}{n \lambda_{th}^3(T_c)} = \frac{Li_{3/2}(z)}{\zeta\left(\frac{3}{2}\right)} \to 1 \quad \text{as } z \to 1$$ [1]

$$\Rightarrow \frac{n_1}{n} = \frac{\lambda_{th}^3(T_c)}{\lambda_{th}^3(T)} = \left(\frac{T}{T_c}\right)^{3/2}$$ [1]

so $$\frac{n_0}{n} = \frac{n - n_1}{n} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$ [1]

Total [5] marks

d) For $T \le T_c$ (or $z \to 1$) one has $N_1 = \frac{V}{\lambda_{th}^3} \zeta\left(\frac{3}{2}\right)$ (as above) [1] and so $\dfrac{V}{\lambda_{th}^3} = \dfrac{N_1}{\zeta(3/2)}$ [1]

Insert this into $U = \frac{3}{2} k_B T \dfrac{V}{\lambda_{th}^3} \zeta(5/2)$ (given in question) [1]

gives $U = \frac{3}{2} N_1 k_B T \dfrac{\zeta(5/2)}{\zeta(3/2)}$ [1]

Now one has from part c) $\dfrac{N_1}{N} = \left(\dfrac{T}{T_c}\right)^{3/2}$ [1]

and so $U = \dfrac{3}{2} \dfrac{\zeta(5/2)}{\zeta(3/2)} N k_B T \left(\dfrac{T}{T_c}\right)^{3/2}$ [1]

Differentiate w.r.t T at constant V and N [1]
gives

$$C_V = \frac{15}{4} \frac{\zeta(5/2)}{\zeta(3/2)} N k_B \left(\frac{T}{T_c}\right)^{3/2}$$ [1]

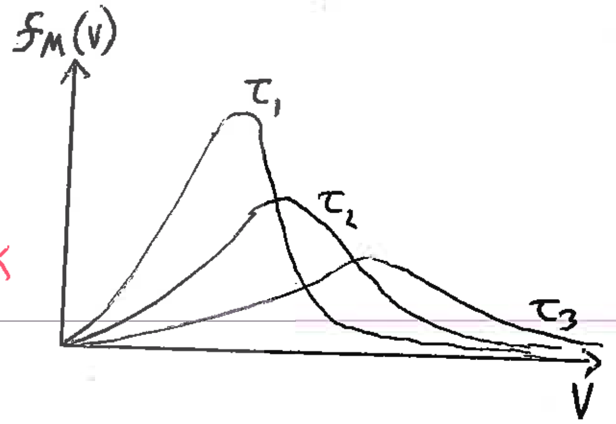
Total [8] marks

Total for B4: [2]+[5]+[5]+[8] = [20] marks

B3)

a)

Bookwork



(with increasing temperature the peak falls and shifts to the right)

[1] + [1] + [1] = [3] marks

The area under each curve is equal to 1 with no units (since $\int_0^\infty f_M(v)\,dv = 1$)

(a probability)

[1] mark

From problem sheet

b) $$\left(\frac{d f_M(v)}{dv}\right)_{v=v_{max}} = 0$$ [1] mark

$$\Rightarrow \left(\frac{d \ln f_M(v)}{dv}\right)_{v=v_{max}} = 0$$ [1] mark

(or use product rule on $f_M(v)$)

so $$\frac{2}{v_{max}} - \frac{m v_{max}}{\tau} = 0 \quad \text{and} \quad v_{max} = \sqrt{\frac{2\tau}{m}}$$ [1]

Total [3] marks

from problem sheet

c) $$\langle v \rangle = \int_0^\infty dv\, v f_M(v) = C \int_0^\infty dv\, v^3 e^{-\frac{mv^2}{2\tau}}$$ where $C = 4\pi \left(\frac{m}{2\pi\tau}\right)^{3/2}$

[1] mark

$$= C\left(\frac{2\tau}{m}\right)^2 \int_0^\infty dx\, x^3 e^{-x^2}$$ where $x = \sqrt{\frac{m}{2\tau}}\, v$

and $dx = \sqrt{\frac{m}{2\tau}}\, dv$

$\boxed{1}$ mark

$$= \frac{C}{2}\left(\frac{2\tau}{m}\right)^2$$ since integral gives factor $\frac{1}{2}$ with $n = a = 1$

$$= \sqrt{\frac{8\tau}{\pi m}}$$ $\boxed{1}$ mark

$$\langle v^2 \rangle = \int_0^\infty dv\, v^2 f_M(v) = C\int_0^\infty dv\, v^4 e^{-\frac{mv^2}{2\tau}}$$ $\boxed{1}$ mark

$$= C\left(\frac{2\tau}{m}\right)^{5/2} \int_0^\infty dx\, x^4 e^{-x^2}$$ $\boxed{1}$ mark

using $x = \sqrt{\frac{m}{2\tau}}\, v$ as above

$$= C\left(\frac{2\tau}{m}\right)^{5/2} \frac{3}{8}\sqrt{\pi}$$ since integral with $n = 2, a = 1$ gives $\frac{3\sqrt{\pi}}{8}$

$$= \frac{3\tau}{m}$$ $\boxed{1}$ mark

and so $v_{rms} = \sqrt{\langle v \rangle^2} = \sqrt{\frac{3\tau}{m}}$ Total $\boxed{6}$ marks

d) Bookwork

$$f_M(v)\,dv = f_M(\epsilon)\,d\epsilon$$ (change of variables from $v \to \epsilon$)

1 mark

since $\epsilon = \frac{1}{2}mv^2$ then $d\epsilon = mv\,dv$ 1 mark

so $$f_M\left(v=\sqrt{\tfrac{2\epsilon}{m}}\right)\frac{d\epsilon}{mv} = f_M(\epsilon)\,d\epsilon$$

and so $$\frac{f_M\left(v=\sqrt{\tfrac{2\epsilon}{m}}\right)}{mv} = f_M(\epsilon)$$ 1 mark

Now $mv = \sqrt{2m\epsilon}$ and write out $f_M\left(v=\sqrt{\tfrac{2\epsilon}{m}}\right)$ explicitly

$$\Rightarrow f_M(\epsilon) = \frac{4\pi}{\sqrt{2m\epsilon}}\left(\frac{m}{2\pi\tau}\right)^{3/2}\frac{2\epsilon}{m}e^{-\frac{\epsilon}{\tau}}$$ 1

$$\Rightarrow f_M(\epsilon) = \sqrt{\epsilon}\,4\pi\left(\frac{1}{\pi\tau}\right)^{3/2}e^{-\epsilon/\tau}\frac{1}{\sqrt{2}}\cdot 2\left(\frac{1}{2}\right)^{3/2}$$

(m cancels out)

$$\Rightarrow f_M(\epsilon) = 2\pi\left(\frac{1}{\pi\tau}\right)^{3/2}\sqrt{\epsilon}\,e^{-\epsilon/\tau}$$ 1

Total [5] marks

e) unseen, but can be deduced from formulae above

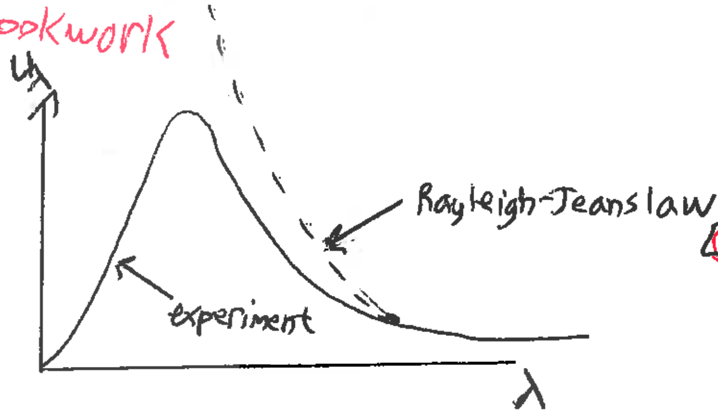
The speed distribution is <u>not</u> ½ the same for the two gases because m is different for each gas ½. The energy distribution <u>is</u> ½ the same because m does not appear in $f_M(\epsilon)$ ½

Total (2)

Total for B1: [4] + [3] + [6] + [5] + [2] = [20] marks

B4) All bookwork

a)



1 + 1 (for each curve 1 mark)

u^{RJ} agrees with experimental functional form for $\lambda \to \infty$ 1 mark, and disagrees for $\lambda \to 0$ 1 mark (ultra-violet catastrophe)

Total [4] marks

b)

Planck assumed that the thermal radiation consists of harmonic oscillators 1 with quantised energy levels 1.

In $\lambda \to \infty$ limit, one has:

$$e^{\frac{\beta hc}{\lambda}} - 1 \to 1 + \frac{\beta hc}{\lambda} - 1 = \frac{\beta hc}{\lambda} \quad 1$$

and so $u_\lambda \to \frac{8\pi k_B T}{\lambda^4}$ 1

In $\lambda \to 0$ limit, $e^{\frac{\beta hc}{\lambda}} \to \infty$ and so $\frac{1}{e^{\frac{\beta hc}{\lambda}} - 1} \to 0$.

This terms dominates over $\frac{1}{\lambda^5}$ and $u_\lambda \to 0$ 1

Total [5] marks

c)

$$u = \int_0^\infty d\lambda \, \frac{8\pi hc}{\lambda^5} \frac{1}{e^{\frac{\beta hc}{\lambda}} - 1}$$ (1) (for correct integration limits)

make substitution $x = \frac{\beta hc}{\lambda}$ (1)

and so $d\lambda = \frac{-\beta hc \, dx}{x^2}$ (1)

so $$u = -\int_\infty^0 dx \frac{\beta hc}{x^2} \frac{8\pi hc x^5}{(\beta hc)^5} \frac{1}{e^x - 1}$$ (1) for integrand and (1) for correct integration limits

$$\Rightarrow u = \frac{8\pi (k_B T)^4}{(hc)^3} I$$ (1) ~~where~~ since $\frac{1}{\beta^4} = (k_B T)^4$

and $I = \int_0^\infty dx \frac{x^3}{e^x - 1}$ (given in question)

note: overall minus sign has disappeared since integration limits are now $\int_0^\infty$ instead of $\int_\infty^0$

Since $I = \frac{\pi^4}{15}$ and $\frac{h}{2\pi} = \hbar$ then $u = \frac{\pi^2}{15} \frac{(k_B T)^4}{(\hbar c)^3}$

[1] mark

Total [7] marks

d)

Thermal radiation may be considered as a gas of photons [1], which obey Bose-Einstein statistics [1]. (This approach gives rise to an identical expression for u_λ.)

A quantised harmonic oscillator (the approach of Planck) in the energy level n [1], is equivalent to n photons, each with energy $\hbar\omega$ [1]

Total [4] marks

Total for B3: [4] + [5] + [7] + [4] = [20] marks