UNIVERSITY OF SOUTHAMPTON   PHYS2024W1

SEMESTER 2 EXAMINATION 2014-2015

QUANTUM PHYSICS OF MATTER

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language word to word® translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. For each of the following thermodynamic variables, state whether it is an extensive or an intensive variable, and whether it is a function of state or not:
i) Temperature (T); ii) Internal energy (U); iii) Entropy (S); iv) Pressure (p).   [4]

A2. An ideal paramagnet has N magnetic dipole moments. Let $n_\uparrow$ and $n_\downarrow$ be the number of spins parallel and antiparallel respectively to an external magnetic field. Write down the statistical weight of each macrostate, $W(N, n_\uparrow)$. Then calculate the statistical entropy σ for the macrostate with $n_\uparrow = n_\downarrow$, giving your answer in terms of N. Show that this entropy is approximately equal to the entropy of all the macrostates combined. You may assume $N, n_\uparrow, n_\downarrow \gg 1$ and use Stirling's formula:

$$\ln(N!) \approx N \ln N - N + \frac{1}{2} \ln(2\pi N) \,.$$ [5]

A3. The Sackur-Tetrode equation for the statistical entropy σ of a classical ideal gas is as follows:

$$\sigma = N \ln \left[\frac{V}{N h^3} e \left(\frac{4\pi m}{3} \right)^{3/2} \left(\frac{E}{N} \right)^{3/2} \right] + \frac{3N}{2} \,.$$

Using the equation

$$\frac{1}{k_B T} = \left(\frac{\partial \sigma}{\partial E} \right)_{V,N} \,,$$

derive an expression for the energy E in terms of the temperature T. Then explain whether or not the Sackur-Tetrode equation agrees with the third law of thermodynamics.   [3]

A4. Find the partition function for a quantum harmonic oscillator in a canonical ensemble with energy levels given by

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right),$$

where n takes positive integer values or is zero. You may use the following result:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad (r < 1).$$

[4]

A5. For a gas of fermions, the Fermi energy (ε_F) is defined as the value of the chemical potential (μ) at $T = 0$. Briefly explain the physical meaning of ε_F.

At $T = 0$, the number of fermions is given by

$$N = \int_{\varepsilon \leq \varepsilon_F} \frac{8g_S\, V\, d^3p}{h^3},$$

where g_S is the spin degeneracy factor, V is volume, and the integration is over the momentum range $p_x > 0$, $p_y > 0$, $p_z > 0$. By introducing the Fermi momentum $p_F = \sqrt{2\,m\,\varepsilon_F}$, where m is the mass of the fermion, show that ε_F is given by

$$\varepsilon_F = \frac{\hbar^2}{2\,m}\left(\frac{6\,\pi^2\,n}{g_S}\right)^{\frac{2}{3}},$$

[4]

where $n = N/V$ and $\hbar = h/2\pi$.

Section B

B1. (a) In the grand canonical ensemble (i.e. a system in thermodynamic equilibrium with a reservoir), the grand potential Φ_G is defined as

$$\Phi_G = U - TS - \mu N.$$

By obtaining an expression for $d\Phi_G$, show that the number of particles N in the system is given by

$$N = -\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{T,V}.$$ [5]

(b) The probability to find the system in a microstate i with energy E_i and number of particles N_i is given by

$$P_i = \frac{1}{\mathcal{Z}} e^{\beta(\mu N_i - E_i)}.$$

Here, the grand-partition function $\mathcal{Z}$ is a sum over all microstates i,

$$\mathcal{Z} = \sum_i e^{\beta(\mu N_i - E_i)},$$

and $\beta = 1/\tau$, where τ is the statistical temperature of the reservoir. Show that

$$\Phi_G = -k_B T \ln \mathcal{Z}.$$ [7]

(Hint: make use of the fact that $\langle N \rangle = \sum_i N_i P_i$.)

(c) For a classical ideal gas in the grand canonical ensemble the grand partition function is given by

$$\mathcal{Z} = \exp\left[e^{\beta \mu} \frac{V}{\lambda_{\text{th}}^3} \right],$$

where the thermal de Brogie wavelength $\lambda_{\text{th}} \propto 1/\sqrt{\beta}$. Calculate Φ_G and then N. Hence obtain the following expression for the chemical potential,

$$\mu = k_B T \ln \left(n \lambda_{\text{th}}^3 \right),$$ [3]

where $n = N/V$.

(d) Using the result that

$$U = - \left(\frac{\partial \ln \mathcal{Z}}{\partial \beta} \right)_{\beta \mu = const} ,$$

show that one obtains the usual expression for the internal energy of a classical ideal gas:

$$U = \frac{3}{2} N k_B T .$$

[5]

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B2. For a gas of bosons the number density of particles (n) can be approximated by the equation

$$n \lambda_{th}^3 = Li_{3/2}(z) .$$

Here $z \equiv e^{\beta \mu}$ is the fugacity and λ_{th} is the thermal de Broglie wavelength. The poly-logarithm function $Li_{3/2}(z)$ is defined by the integral

$$Li_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{dx\, x^{1/2}}{z^{-1} e^x - 1} ,$$

where $x = \beta \varepsilon$ and ε is energy.

(a) Explain the physical meaning of the thermal de Broglie wavelength, which is defined as

$$\lambda_{th}(T) = \hbar \sqrt{\frac{2\pi}{g_S^{2/3} m k_B T}} ,$$

where g_S is the spin degeneracy of the bosons in the gas, m is the mass of a boson, and T is temperature. [2]

(b) In the low temperature limit ($z \to 1$) one has $Li_{3/2}(z) \to \zeta(3/2) \approx 2.6$. Hence show that the equation $n \lambda_{th}^3 = Li_{3/2}(z)$ cannot be solved for z for all values of n and T in this limit of $z \to 1$. Explain briefly the reasons why this equation cannot be solved (i.e. the approximation has broken down). [5]

(c) The critical temperature (T_c) is defined by the equation

$$n \lambda_{th}^3(T = T_c) = \zeta(3/2) .$$

Show that

$$\frac{n_0}{n} = \frac{n - n_1}{n} = 1 - \left(\frac{T}{T_c} \right)^{3/2} ,$$

where n_0 is the number density of particles in the g_S ground states of energy, and n_1 is the number density of particles in all other energy states. [5]

(d) The internal energy (U) of a gas of bosons for $T \leq T_c$ is given by the expression

$$U = \frac{3}{2} k_B T \frac{V}{\lambda_{th}^3} \zeta(5/2) \,,$$

where V is the volume of the gas. Show that the heat capacity at constant volume (C_V) for $T \leq T_c$ is given by

$$C_V = \frac{15}{4} \frac{\zeta(5/2)}{\zeta(3/2)} N k_B \left(\frac{T}{T_c} \right)^{3/2} .$$

[8]

B3. The Maxwell distribution of speeds (v) for the particles of a classical ideal gas is given by

$$f_M(v) = 4 \pi \left(\frac{m}{2 \pi \tau} \right)^{3/2} v^2 e^{-\frac{m v^2}{2 \tau}} \,,$$

where m is the mass of a gas particle and τ is the statistical temperature.

(a) Qualitatively sketch $f_M(v)$ for three different temperatures $\tau_1 < \tau_2 < \tau_3$, and state the value of the area (giving the units) under each curve. [4]

(b) The maximum value of $f_M(v)$ occurs at $v = v_{\max}$. Derive an expression for $v_{\max}$ as a function of m and τ. [3]

(c) Derive expressions for the expectation value $\langle v \rangle$ and root mean squared speed $v_{\mathrm{rms}} = \sqrt{\langle v^2 \rangle}$. You should use the following integrals:

$$\int_0^\infty dx \, x^{2n+1} e^{-a x^2} = \frac{n!}{2 \, a^{n+1}} \quad \text{and} \quad \int_0^\infty dx \, x^{2n} e^{-a x^2} = \frac{(2n)!}{n! \, 2^{2n+1}} \sqrt{\frac{\pi}{a^{2n+1}}} \,.$$

[6]

(d) Now show that the energy distribution of a gas particle is given by

$$f_M(\varepsilon) = 2 \pi \left(\frac{1}{\pi \tau} \right)^{3/2} \sqrt{\varepsilon} \, e^{-\frac{\varepsilon}{\tau}} \,,$$

where ε is the energy of the particle. [5]

(e) For two different ideal gases (e.g. oxygen and nitrogen) at the same temperature, very briefly explain i) whether or not the speed distribution is the same for the two gases, and ii) whether or not the energy distribution is the same for the two gases.   [2]

B4. (a) In classical physics the spectral energy density in terms of wavelength (u_λ^{RJ}) for thermal radiation (which is in equilibrium with the walls of a cavity) is called the "Rayleigh-Jeans" law. It is given by the expression

$$u_\lambda^{RJ} = \frac{8\pi}{\lambda^4} k_B T \,,$$

where T is temperature. Qualitatively plot this expression for u_λ^{RJ} as a function of λ for a fixed temperature, and on the same figure draw the functional form for u_λ which is derived from experiment. In which limit (either $\lambda \to 0$ or $\lambda \to \infty$) does u_λ^{RJ} agree with the experimental functional form, and in which limit does it disagree? [4]

(b) The Planck distribution for u_λ is given by the expression

$$u_\lambda = \frac{8\pi}{\lambda^5} \frac{hc}{e^{\frac{\beta hc}{\lambda}} - 1} \,.$$

What assumption did Planck make for the thermal radiation in order to obtain this expression for u_λ? Derive the behaviour of u_λ when λ is very large and when λ is very small. [5]

(c) By integrating u_λ in (b), show that the total energy density u is given by

$$u = \frac{\pi^2}{15} \frac{(k_B T)^4}{(\hbar c)^3} \,.$$

[Note: Use the integral

$$\int_0^\infty dx \, \frac{x^3}{e^x - 1} = \frac{\pi^4}{15} \,. \,]$$

[7]

(d) Qualitatively, briefly discuss an alternative way of obtaining the Planck distribution u_λ which makes a different assumption about the nature of thermal radiation, and comment on the equivalence of the two approaches. [4]

END OF PAPER