

SEMESTER 2 EXAMINATION 2012-2013

Quantum Physics of Matter

Duration: 120 MINS (2 hours)

This paper contains 8 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries $1/3$ of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries $2/3$ of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

- A1.** Write down the Boltzmann formula that defines the statistical entropy σ for an isolated system in equilibrium, explaining the meaning of the quantities involved. What is the dimensionality of σ ? How is the statistical entropy σ related to the thermodynamic entropy S ? [4]
- A2.** (a) State the postulate of equal a priori probabilities and explain how from this it is possible to determine the statistical distribution within the microcanonical ensemble.
- (b) Write down the statistical distribution $\rho(p, q)$ for a microcanonical ensemble (microcanonical distribution), for a classical system described by a set of (generalised) momenta and coordinates (p, q) motivating its expression (you can indicate the normalisation constant generically with C and leave it unspecified). [5]
- A3.** Classical statistical mechanics does not unambiguously reproduce the correct expression for the entropy of an ideal gas of identical particles (Sackur-Tetrode formula) in agreement with the laws of thermodynamics for two reasons: discuss them briefly (in no more than 150 words). [5]
- A4.** Write down the Fermi-Dirac distribution function for a quantum ideal gas, explaining its physical meaning. In what limit does a (non-relativistic) Fermi gas become degenerate for a fixed particle number density? Specify in this case the asymptotic limit of the Fermi-Dirac distribution function discussing, in no more than 100 words, its physical meaning. [6]

Section B

- B1.** (a) Consider a macroscopic system. What is the definition of the term microstate? What is the definition of the term macrostate? What is the definition of statistical weight of a macrostate? What is the definition of phase space for a classical system? And for a discrete system? [5]
- (b) Consider a classical system described by a set of (generalised) momenta and coordinates $(p, q) \equiv (p_1, p_2, \dots, p_M; q_1, q_2, \dots, q_M)$, where M is the number of degrees of freedom. Consider an ideal gas of N identical particles. What is the number of degrees of freedom M in this case? What is the total energy $E(p, q)$ of such a system? [4]
- (c) What is the number of microstates in the infinitesimal volume of the phase space $dp dq$ if the particles are assumed to be distinguishable? How does it change if the particles are assumed to be indistinguishable and why? [4]
- (d) Consider an isolated ideal paramagnet made of 5 particles with dipole magnetic moment \vec{m} that can have only two discrete values: either $\vec{m} = +m \hat{z}$ or $\vec{m} = -m \hat{z}$ ($m \equiv |\vec{m}|$). What specifies a macrostate? What is the total number of microstates? Assume the equal a priori probability postulate. What is the probability of that particular microstate where all five dipole magnetic moments are positive? What is the probability of the microstate where all five magnetic moments are negative? What is the statistical weight and the probability of the macrostate where 3 particles have positive magnetic moment? [7]

TURN OVER

- B2.** (a) Write down the fundamental relation of thermodynamics for a macroscopic system with a fixed number of particles explaining the meaning of the quantities involved. What is the meaning of the fundamental relation of thermodynamics? How does it get specialised in the case of an isolated system? How is it related to the laws of thermodynamics? [5]
- (b) Starting from the fundamental relation of thermodynamics, derive the exact differential for the free energy F properly defined in terms of the other thermodynamic quantities. How can the pressure be calculated from the free energy? [4]
- (c) Consider the canonical ensemble. How is it defined? Write down the canonical distribution and give the definition of partition function. [4]
- (d) The partition function for an ideal gas of N non-relativistic identical particles with mass m , in a volume V at temperature T is given by

$$Z = \frac{1}{N!} \left(\frac{V}{\lambda_{\text{th}}^3} \right)^N,$$

where

$$\lambda_{\text{th}} \equiv \frac{h}{\sqrt{2\pi m \tau}},$$

is the thermal De Broglie wavelength. How is the statistical temperature τ related to the thermodynamic temperature T ? Show that the free energy F is given by

$$F = -N \tau \log \left[\frac{e V}{N \lambda_{\text{th}}^3} \right]. \quad [4]$$

- (e) Derive the equation of state for an ideal gas from the free energy F . [3]

- B3.** (a) Show that the Bose-Einstein distribution for the mean occupation numbers $\langle n_k \rangle$ of the one-particle quantum states of a gas of (non-interacting) bosons is given by

$$\langle n_k \rangle = \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1},$$

where $\beta \equiv 1/(k_B T)$, ε_k is the energy of the state k and μ is the chemical potential.

Hint: calculate the grand-potential $\Phi_G(E_k)$ for the particles in a quantum state with energy $E_k = n_k \varepsilon_k$ (system with a variable number of particles) and then use

$$\langle n_k \rangle = -\frac{\partial \Phi_G(E_k)}{\partial \mu}.$$

[6]

- (b) Consider now the Bose-Einstein distribution function for a gas of photons (Planck distribution function). Express it in terms of the photon wavelength λ and derive the Planck law for the spectral energy density u_λ of the black body radiation given by

$$u_\lambda = \frac{8 \pi h c}{\lambda^5} \frac{1}{e^{\frac{\beta h c}{\lambda}} - 1}.$$

Hint: The number density of photon states in the infinitesimal interval $[p, p + dp]$ is given by:

$$2 \frac{dp p^2}{2 \pi^2 \hbar^3},$$

where the pre-factor 2 is the number of polarizations.

[4]

- (c) Derive the Rayleigh-Jeans law in the limit of long wavelengths, when $\beta h c / \lambda \ll 1$. Explain why the Planck constant disappears and what the ultraviolet catastrophe is.

[6]

- (d) By integrating the spectral energy density u_λ , derive the expression for the

TURN OVER

total energy density

$$u = \frac{\pi^2}{15} \frac{k_B^4 T^4}{\hbar^3 c^3},$$

using

$$\int_0^{\infty} dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}.$$

[4]

B4. Consider a crystalline solid where N atoms, with $N \gg 1$, are arranged in a lattice, such that their mean positions form a three-dimensional array of regularly spaced points.

(a) How many vibrational modes does such a system have ?

Deduce the heat capacity from the theorem of equipartition of energy (Dulong and Petit's Law). [3]

(b) Consider now the Einstein model, where the vibrational modes are described in terms of quantum harmonic oscillators with the same angular frequency ω_E . Show that the heat capacity is given by

$$C = 3 N k_B \frac{y^2 e^y}{(e^y - 1)^2},$$

where $y = \Theta_E/T$ and $\Theta_E = \hbar\omega_E/k_B$.

Hint: start calculating the mean energy of one quantum harmonic oscillator with angular frequency ω_E and use $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$, for $r < 1$. [6]

(c) Explain the assumptions of the Debye model, in particular how the Debye's frequency is defined, and how it differs from the Einstein model. [5]

(d) In the Debye model the heat capacity is given by

$$C = \frac{9 N k_B}{x_D^3} \int_0^{x_D} dx \frac{x^4 e^x}{(e^x - 1)^2},$$

where $x_D = \Theta_D/T$ and $\Theta_D = \hbar\omega_D/k_B$ is the Debye temperature. Show that in the limit of low temperatures ($x_D \gg 1$) the heat capacity behaves as $C \propto T^3$. How would this behavior be modified for a solid described by a two-dimensional lattice? [3]

(e) Discuss, in no more than 150 words, the physical meaning of phonons and how they can be used to describe lattice vibrations. In particular, explain similarities and differences between photons and phonons. [3]

END OF PAPER