SEMESTER 2 EXAMINATION 2013-2014

THEORIES OF MATTER, SPACE AND TIME

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

The Lorentz transformation matrix for a boost along the positive x direction by speed v is

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Electric and magnetic field components may be written as the elements of the tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}.$$

The metric tensor in our convention is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Throughout the paper we assume the standard notation of c as the speed of light in vacuum, and h as Planck's constant.

Section A

A1. Show that the Lagrangian describing the motion of a projectile of mass m in the gravitational field close to the Earth's surface is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

where y is the height above the Earth's surface and x is a direction along the surface. [1]

Write down and evaluate the Euler-Lagrange equations for the system. [2]

Show that there are two conserved quantities and find expressions for them. [2]

A2. A photon of frequency f moving in the negative x direction has the relativistic four momentum components,

$$p^\mu = (\frac{E}{c}, \vec{p}) = (\frac{hf}{c}, -\frac{hf}{c}, 0, 0).$$

Show that, under a boost by speed v in the positive x direction, the frequency of the photon is Doppler shifted to,

$$f' = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}\, f.$$

[3]

A3. The four-velocity of a particle is defined in relativity by

$$u^\mu = \frac{dx^\mu}{d\tau},$$

where x^μ is the four-vector describing the space-time position of the particle and τ is the proper time. Derive an expression for the components of u^μ in terms of the components of velocity observed in some arbitrary inertial frame. [3]

Hence show that

$$u^\mu u_\mu = c^2.$$

[1]

TURN OVER

A4. Find the x component of the Lorentz force on a point charge q in the presence of an electric field $\vec{E}$ and a magnetic field $\vec{B}$,

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}).$$

[2]

Show that this matches the x component of the Lorentz covariant force,

$$f^{\mu} = q u_{\nu} F^{\mu\nu},$$

in the non-relativistic limit. [2]

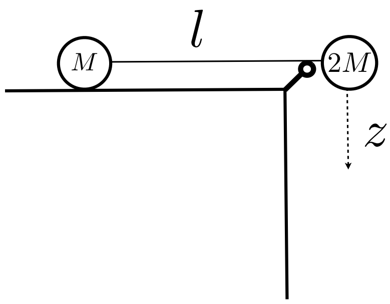
A5. Starting with the relativistic energy-momentum relation

$$E^2 = m^2 c^4 + \vec{p}^2 c^2,$$

where m is the mass of the particle, $\vec{p}$ is its three-momentum and E its energy, write down the energy and momentum as differential operators and obtain the Klein-Gordon equation involving the $\Box$ operator and the wavefunction ϕ. [4]

Section B

B1. (a) Two balls of mass M and $2M$ are attached by a massless inextensible string of length l via a frictionless pulley. The second ball falls from rest under gravity, starting from level with the table, with the first ball being pulled without rotation across the frictionless table, as shown in the figure.



Using the height below the table, z, as the generalised coordinate, write down the Lagrangian for this system valid for $z \leq l$. [4]

(b) Find the Euler-Lagrange equation for this system and hence find the acceleration of the falling ball. [3]

(c) Compute the time taken for the first ball to reach the edge of the table. [2]

(d) If the massless string is replaced by a heavy string with mass m distributed uniformly along it, find the Euler-Lagrange equation in this case. [4]

Hence show that the acceleration of the falling ball with a heavy string as a function of $z \leq l$ can be written as $\ddot{z} = az + b$ and find a and b. [2]

Verify that the solution to $\ddot{z} = az + b$ has the form

$$z(t) = -\frac{b}{a} + c_1 e^{\sqrt{a}t} + c_2 e^{-\sqrt{a}t}.$$

[2]

If the second ball falls from rest as in the above diagram, find the constants c_1 and c_2 and hence find $z(t)$ in the case of the heavy string. [3]

TURN OVER

B2. (a) By defining the four-momentum of a particle p^μ in terms of its energy E and its three-momentum $\vec{p}$, show how E and $\vec{p}$ transform under a Lorentz boost along the positive x direction by speed v. [4]

(b) A beam of antiprotons in the laboratory moving at speed u in the positive x direction is directed onto a lead target containing static protons. Write the four-momentum components of an antiproton p_a^μ and a proton p_b^μ in the laboratory frame, then perform a boost by a speed v in the positive x direction to find their components in the centre of mass frame. [4]

Hence show that to move to the centre of mass frame for a collision between one of the anti-protons and a proton in a lead nucleus a boost must be made by speed

$$v = \frac{\gamma(u)u}{1+\gamma(u)}.$$

[4]

(c) Consider a charged pion of mass m_π produced at rest in the centre of mass frame, which subsequently decays into a neutrino with zero mass and a muon of mass m_μ.

Working in the centre of mass frame, write down expressions for the components of the four-momenta of the pion p_0^μ and its decay products, the neutrino p_1^μ and muon p_2^μ. [2]

Using conservation of 4-momentum express p_1^μ in terms of p_0^μ and p_2^μ. Hence, by considering $p_1^\mu p_{1\mu}$, find expressions for the energy of the neutrino and the muon in terms of m_π and m_μ. [6]

B3. (a) By performing a boost with speed v in the positive x direction on $F^{\mu\nu}$, show that the electric field components transform as

$$\frac{E'_x}{c} = \frac{E_x}{c}$$
$$\frac{E'_y}{c} = \gamma\left(\frac{E_y}{c} - \frac{v}{c}B_z\right)$$
$$\frac{E'_z}{c} = \gamma\left(\frac{E_z}{c} + \frac{v}{c}B_y\right).$$

[10]

(b) Two of Maxwell's equations can be written in covariant form as,

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu,$$

where $J^\nu = (\rho c, \vec{J})$.

By applying this equation for $\nu = 0$ and $\nu = 1$, determine the two Maxwell equations for the specified components. [8]

(c) By introducing the four-vector potential A^μ where,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

show that in Lorenz gauge,

$$\Box A^\nu = \mu_0 J^\nu.$$

[2]

TURN OVER

B4. The Schroedinger equation for the wave function ψ describing a non-relativistic particle with mass m, moving in a potential V is given by

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi.$$

If the potential is time independent there are solutions of the form

$$\psi = u(\vec{x}) e^{-iEt/\hbar},$$

with u satisfying the time independent Schroedinger equation

$$Eu = -\frac{\hbar^2}{2m} \nabla^2 u + Vu.$$

(a) Show that the probability density $\rho = \psi^* \psi$ satisfies an equation of the form

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{J} = 0$$

where $\vec{J}$ is the probability density current. Explain the physical meaning of this equation. [7]

(b) Compute both ρ and $\vec{J}$ for the case of a free particle with energy E and momentum $\vec{p}$. [3]

(c) Suppose that for a particular potential $V_0(\vec{x})$ the full set of solutions $u_n(\vec{x})$ with energy eigenvalues E_n are known. Explain with words and equations what it means for the set of eigenfunctions $u_n(\vec{x})$ to be both complete and orthonormal. [4]

(d) In the scenario in (c) the wave function, $\psi(\vec{x}, t)$, can be prepared at time $t = 0$ to take an arbitrary form $\phi(\vec{x})$. Explain in detail how the time evolution of this wave function can, in principle, be determined using the known energy eigenstates $u_n(\vec{x})$. [6]

END OF PAPER

