Phys 3007 - 2015

Theories of matter, space and time

Model answers

[lecture notes]

A1

i) $L(q_1 \ldots q_n, \dot{q}_1 \ldots \dot{q}_n, t)$ Lagrangian

q_k ignorable, or cyclic, if L does not depend on q_k [1]

in which case $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = 0$

$\Rightarrow p_k = \frac{\partial L}{\partial \dot{q}_k}$ conserved [1]

ii) Lagrangian for a particle moving in a plane under a central conservative force:

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - V(r) \quad [1]$$

L independent of θ - θ is cyclic $\Rightarrow$

$$\Rightarrow 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}(\underbrace{m r^2 \dot{\theta}}_{\ell = \text{angular momentum}}) \quad [1]$$

is conserved

[Variant of example in lectures]

A2

$\Delta x^{\mu} \Delta x_{\mu}$ invariant

$$\Rightarrow \quad c^2 \Delta t_{(2)}^2 - \Delta \underline{x}_{(2)}^2 = c^2 \Delta t_{(1)}^2 - \Delta \underline{x}_{(1)}^2 \qquad [1]$$

$$\Delta \underline{x}_{(1)} = 0 \quad \Rightarrow \quad \Delta x_{(2)} = c\sqrt{\Delta t_{(2)}^2 - \Delta t_{(1)}^2}$$

$$= 3\,10^8 \sqrt{16-4} \ \text{m} \qquad [1]$$

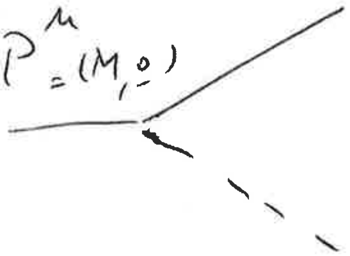
$$\simeq 10^9 \ \text{m}$$

[variant of question in problem sheet]

A3]

$p^\mu = (E, \underline{p})$ $(p^2 = m^2)$

$P^\mu = (M, \underline{0})$



p'^μ $(p'^2 = 0)$

Four-momentum conservation implies

$$P - p = p' \Rightarrow (P-p)^2 = p'^2 \Rightarrow M^2 + m^2 - 2mE = 0 \quad [1]$$

$$\Rightarrow \quad E = \frac{M^2 + m^2}{2M}$$

Therefore

$$T = E - m = \frac{M^2 + m^2}{2M} - m = \frac{(M-m)^2}{2M} \quad [1]$$

[redacted]

A4)

[lecture notes]

a) Four-gradient $\partial^\mu = \left(\frac{\partial}{\partial t}, -\underline{\nabla}\right)$ [1]

Four-potential $A^\mu = (\varphi, \underline{A})$ [1]

Four-current $J^\mu = (\rho, \underline{J})$ [1]

Field strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ [1]

b) The two terms on the lhs of the given equations are respectively $\partial_\mu \partial^\mu A^\nu$ and $-\partial^\nu \partial_\mu A^\mu$, while the rhs is $\mu_0 J^\nu$ $(c = 1 \Rightarrow \mu_0 \varepsilon_0 = 1)$:

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_0 J^\nu \quad [1]$$

By exchanging order of derivatives in the 2nd term on the lhs we have

$$\partial_\mu \underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{F^{\mu\nu}} = \mu_0 J^\nu \quad [1]$$

i.e.

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$$

[redacted]

[lecture]

A5)

Multiply Klein-Gordon equation by ϕ^*

$$\phi^* \partial_\mu \partial^\mu \phi + m^2 \phi^* \phi = 0 \qquad [1]$$

and take complex conjugate equation multiplied by ϕ

$$\phi \partial_\mu \partial^\mu \phi^* + m^2 \phi^* \phi = 0 \qquad [1]$$

subtract the two equations: mass terms cancel, and we get

$$\phi^* \partial_\mu \partial^\mu \phi - \phi \partial_\mu \partial^\mu \phi^* = 0 \qquad [1]$$

that is,

$$\partial_\mu (\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*) - (\partial_\mu \phi^*)(\partial^\mu \phi) + (\partial_\mu \phi)(\partial^\mu \phi^*) = 0$$

[1]

Cancelling the last two terms,

$$\partial_\mu \underbrace{(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)}_{-i j^\mu} = 0 \qquad [1]$$

$$\Rightarrow \quad j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*), \quad \partial_\mu j^\mu = 0 \qquad [1]$$

[lecture example]

B1

a) upper mass: $L_1 = \frac{1}{2} m_1 \ell^2 \dot\vartheta^2 - m_1 g \ell (1-\cos\vartheta)$ [2]

lower mass: $L_2 = \frac{1}{2} m_2 \ell^2 \left(\dot\vartheta^2 + \dot\phi^2 + 2\cos(\phi-\vartheta)\dot\vartheta\dot\phi \right)$

$- m_2 g \ell (1-\cos\vartheta) - m_2 g \ell (1-\cos\phi)$ [2]

$L = L_1 + L_2$ [2]

b) $m_1 = m_2 = m$. For small oscillations,

$$L = \frac{1}{2} m \ell^2 \left(2\dot\vartheta^2 + \dot\phi^2 + 2\dot\vartheta\dot\phi \right) - mg\ell\left(\vartheta^2 + \frac{\phi^2}{2}\right) \quad [2]$$

Euler Lagrange equations:

$$\begin{cases} 2m\ell^2\ddot\vartheta + m\ell^2\ddot\phi = -2mg\ell\vartheta & [1] \\ m\ell^2\ddot\phi + m\ell^2\ddot\vartheta = -mg\ell\phi & [1] \end{cases}$$

i.e.

$$\begin{cases} 2\ddot\vartheta + \ddot\phi = -2\frac{g}{\ell}\vartheta \\ \ddot\phi + \ddot\vartheta = -\frac{g}{\ell}\phi \end{cases}$$

Separating second derivative terms,

$$\ddot\vartheta = -g/\ell\,(2\vartheta - \phi) \quad [1]$$

$$\ddot\phi = -\frac{2g}{\ell}(\phi - \vartheta) \quad [1]$$

c) $$\frac{d^2}{dt^2}\begin{pmatrix}\vartheta\\ \phi\end{pmatrix} = -\underbrace{\begin{pmatrix}2g/\ell & -g/\ell\\ -2g/\ell & 2g/\ell\end{pmatrix}}_{\vartheta}\begin{pmatrix}\vartheta\\ \phi\end{pmatrix}$$

Normal frequencies:

$$\det(\vartheta - \omega^2 \mathbb{1}) = 0 \Rightarrow \det\begin{pmatrix}2g/\ell - \omega^2 & -g/\ell\\ -2g/\ell & 2g/\ell - \omega^2\end{pmatrix} = 0 \qquad [2]$$

i.e., $$\left(\frac{2g}{\ell} - \omega^2\right)^2 - 2\left(g/\ell\right)^2 = 0$$

So $$\omega_\pm^2 = \frac{2g}{\ell} \pm \sqrt{2}\, g/\ell = g/\ell\,(2 \pm \sqrt{2}) \qquad [2]$$

$$\frac{\omega_+}{\omega_-} = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = \frac{2+\sqrt{2}}{\sqrt{2}} = \sqrt{2}+1 \qquad [1]$$

d) For the two normal modes, $\begin{pmatrix}\vartheta\\ \phi\end{pmatrix} = \begin{pmatrix}\Theta_\pm\\ \Phi_\pm\end{pmatrix} e^{i\omega_\pm t}$

$$\Rightarrow \begin{pmatrix}-\omega_\pm^2 + 2g/\ell & -g/\ell\\ -\frac{2g}{\ell} & -\omega_\pm^2 + 2g/\ell\end{pmatrix}\begin{pmatrix}\Theta_\pm\\ \Phi_\pm\end{pmatrix} = 0 \qquad [1]$$

So $$(-\omega_\pm^2 + 2g/\ell)\,\Theta_\pm - g/\ell\,\Phi_\pm = 0$$

that is, $$\mp\sqrt{2}\, g/\ell\,\Theta_\pm - g/\ell\,\Phi_\pm = 0 \qquad [1]$$

$$\Rightarrow \Phi_\pm = \mp\sqrt{2}\,\Theta_\pm \qquad [1]$$

[lecture notes + variant of example]

B2

a) 4-momentum conservation $p' = p + k - k'$ [2]

$$\Rightarrow \quad m^2 = p'^2 = (p+k-k')^2 = m^2 + 2m(\omega - \omega') + $$
$$- 2\omega\omega'(1-\cos\vartheta) \qquad [2]$$

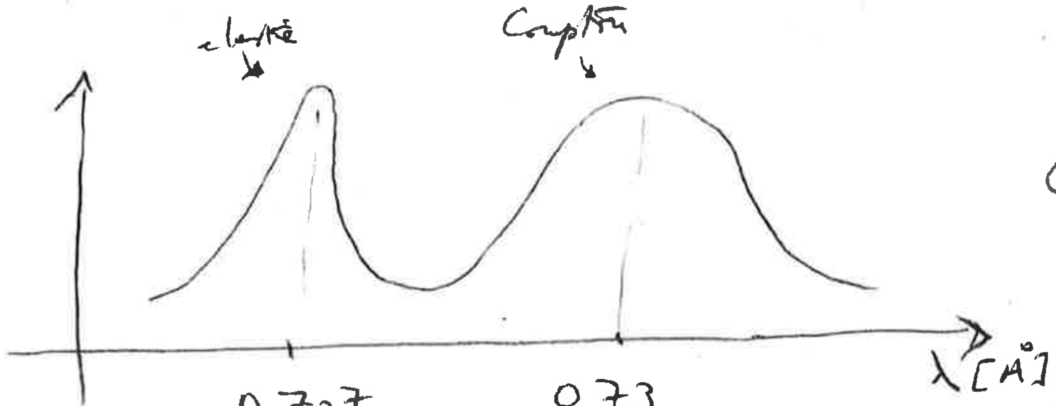
i.e.

$$2m\omega = 2\omega'\left[m + \omega(1-\cos\vartheta)\right]$$

so
$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\vartheta)} \qquad [2]$$

b) elastic scattering by nuclei and tightly bound electrons [1]

Compton scattering off nearly free electrons [1]



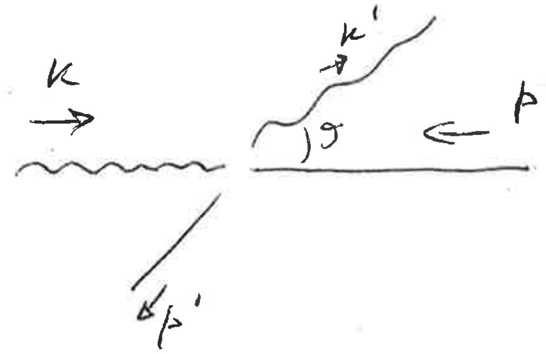
[2]

[2]

↗ 17.5 KeV

↑ 16.95 KeV (≈3% Compton shift at 90°)

e)



Applying 4-momentum conservation now gives

$$m^2 = p'^2 = (p+k-k')^2 = m^2 + \underbrace{2p\cdot(k-k')}_{2E_e(E-E') - 2\sqrt{E_e^2-m_e^2} * (-E + E'\cos\vartheta)} - 2EE'(1-\cos\vartheta) \quad [1]$$

$$\Rightarrow 2EE'(1-\cos\vartheta) = 2E_e(E-E') + 2\sqrt{E_e^2-m_e^2}\,(E - E'\cos\vartheta) \quad [1]$$

that is,

$$E'\left[2E(1-\cos\vartheta) + 2E_e + 2\sqrt{E_e^2-m_e^2}\cos\vartheta\right] = 2E_eE + 2\sqrt{E_e^2-m_e^2}\,E$$

So
$$E' = \frac{E\left(E_e + \sqrt{E_e^2 - m_e^2}\right)}{\sqrt{E_e^2-m_e^2}\cos\vartheta + E_e + E(1-\cos\vartheta)} \quad [1]$$

Maximum E' for minimum of denominator $\Rightarrow \vartheta = \pi$ [1]

Expanding $\sqrt{E_e^2 - m_e^2} \simeq E_e - \frac{m_e^2}{2E_e}$ gives [1]

$$E'_{max} \simeq \frac{2EE_e}{2E + \cancel{E_e} - \cancel{E_e} + \frac{m_e^2}{2E_e}} = \frac{E}{E/E_e + \frac{m_e^2}{4E_e^2}} \quad [1]$$

$\lambda = 514 \Rightarrow E = 2.4\,\mathrm{eV}$

Thus
$$E'_{max} \simeq \frac{2.4}{\frac{2.4}{27\,10^9} + \frac{0.511^2\,10^{12}}{4\,27^2\,10^{18}}}\,\mathrm{eV} \simeq 13.5\,\mathrm{GeV}$$
$\Rightarrow 5\,10^9$ increase [2]

[lectures]

B3

$$L = \frac{1}{2} m \dot{x}^2 - e\varphi + e \dot{\underline{x}} \cdot \underline{A}$$

a) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_j} - \frac{\partial L}{\partial x_j} = 0 \Rightarrow \qquad \nabla_j \equiv \frac{\partial}{\partial x_j}$$

$$\Rightarrow \frac{d}{dt}(m \dot{x}_j + e A_j) + e \nabla_j \varphi - e \nabla_j (\dot{\underline{x}} \cdot \underline{A}) = 0 \qquad [1]$$

using $$\frac{d}{dt} A_j = \frac{\partial}{\partial t} A_j + \dot{\underline{x}} \cdot \underline{\nabla} A_j \Rightarrow \qquad [1]$$

$$\Rightarrow m \ddot{x}_j + e \frac{\partial A_j}{\partial t} + e \dot{\underline{x}} \cdot \underline{\nabla} A_j + e \nabla_j \varphi - e \nabla_j \dot{\underline{x}} \cdot \underline{A} = 0 \qquad [1]$$

$-e E_j$

$(\underline{E} = -\underline{\nabla}\varphi - \frac{\partial \underline{A}}{\partial t})$

$-e(\dot{\underline{x}} \wedge \underline{B})_j$ [2]

because

$$\dot{\underline{x}} \wedge (\underline{\nabla} \wedge \underline{A})_j = \nabla_j (\dot{\underline{x}} \cdot \underline{A}) - (\dot{\underline{x}} \cdot \underline{\nabla}) A_j$$

$(\underline{B} = \underline{\nabla} \wedge \underline{A})$

Thus

$$m \ddot{x}_j = e \left[E_j + (\dot{\underline{x}} \wedge \underline{B})_j \right]$$

$$m \ddot{\underline{x}} = e \left[\underline{E} + \dot{\underline{x}} \wedge \underline{B} \right] \qquad [2]$$

b) $$p_j = \frac{\partial L}{\partial \dot{x}_j} = m \dot{x}_j + e A_j \qquad [1]$$

depends on vector potential   [1]

c) $$H = \sum_j p_j \dot{x}_j - L$$

$$= \underline{p}\cdot\dot{\underline{x}} - \frac{1}{2}m\dot{\underline{x}}^2 + e\varphi - e\dot{\underline{x}}\cdot\underline{A} \qquad [1]$$

using $\underline{p} = m\dot{\underline{x}} + e\underline{A} \Rightarrow \dot{\underline{x}} = \frac{1}{m}(\underline{p} - e\underline{A})$ [1]

$$= \frac{1}{m}\underline{p}\cdot(\underline{p} - e\underline{A}) - \frac{1}{2m^2}m(\underline{p} - e\underline{A})^2 + e\varphi - \frac{e}{m}(\underline{p} - e\underline{A})\cdot\underline{A} \qquad [2]$$

$$\frac{1}{m}(\underline{p} - e\underline{A})^2$$

$$= \frac{1}{2m}(\underline{p} - e\underline{A})^2 + e\varphi \qquad [2]$$

d) $L \to L + \frac{df}{dt}$, $\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\underline{x}}\cdot\underline{\nabla}f$ has no effect on equations of motion [1]

It is equivalent to gauge transformations of the potentials: [1]

$$L + \frac{df}{dt} = \frac{1}{2}m\dot{\underline{x}}^2 - e\varphi + \frac{\partial f}{\partial t} + e\dot{\underline{x}}\cdot\underline{A} + \dot{\underline{x}}\cdot\underline{\nabla}f$$

$$= \frac{1}{2}m\dot{\underline{x}}^2 - e\underbrace{\left[\varphi - \frac{\partial}{\partial t}(f/e)\right]}_{\varphi'} + e\dot{\underline{x}}\cdot\underbrace{\left[\underline{A} + \underline{\nabla}(f/e)\right]}_{\underline{A}'} \qquad [2]$$

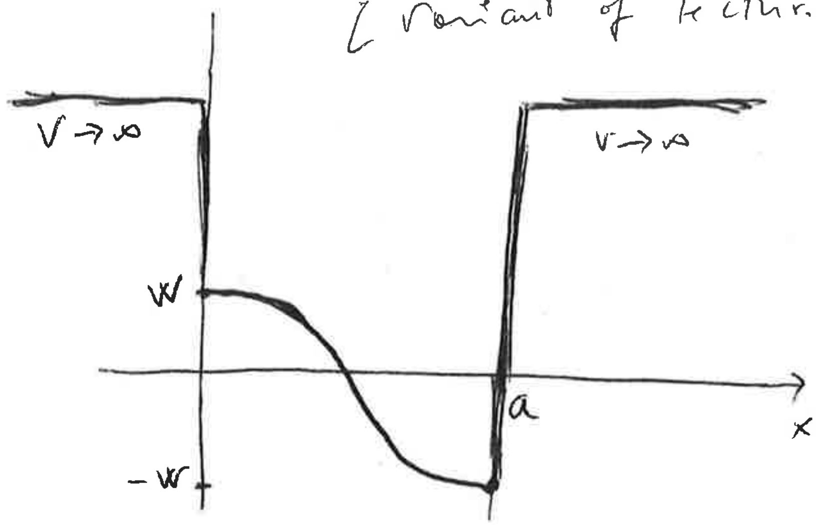
gauge-transformed potentials

$\varphi' = \varphi - \partial\lambda/\partial t$, $\underline{A}' = \underline{A} + \underline{\nabla}\lambda$ with $\lambda = f/e$ (1)

[variant of lecture example]

B4)

a)



[2]

b) The stationary states of the unperturbed system have well-defined parity about $x = \frac{a}{2}$ [1]
while the perturbation has odd parity [1]

Therefore $\delta E_n^{(1)} = \int u_n^* H_1 u_n \, dx = 0$ for all n. [2]

c) $$\delta \psi_1^{(1)} = \sum_m u_m \frac{1}{E_1^{(0)} - E_m^{(0)}} \int u_m^* H_1 u_1 \quad [1]$$

Note that $H_1 u_1 = W \cos\frac{\pi x}{a} \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a} =$
$$= \frac{W}{2} \sqrt{\frac{2}{a}} \sin\frac{2\pi x}{a} = \frac{W}{2} u_2 \quad [1]$$

Thus $\int u_m^* H_1 u_1 = \frac{W}{2} \delta_{m2}$ [1]

$$\Rightarrow \delta \psi_1^{(1)} = \frac{W}{2} \frac{1}{E_1^{(0)} - E_2^{(0)}} u_2 = - \frac{W}{6 E_1^{(0)}} u_2 \quad [2]$$

$$\psi_1^{(1)} = u_1 - \frac{W}{6 E_1^{(0)}} u_2$$

The correction tends to cancel the zeroth-order [1]

Contribution for $x < \frac{a}{2}$ and add to it for $x > \frac{a}{2}$ [1]

$\Rightarrow$ particle is more localised around the minimum of the potential. [1]

Perturbation theory gives a sensible approximation provided the coefficient in $\delta\psi_1^{(1)}$ is small, i.e. $W \ll 6E_1^{(0)}$. [2]

d) $$\delta E_n^{(2)} = \sum_m \frac{\left|\int u_m^* H_1 u_n\right|^2}{E_n - E_m}$$ [1]

$n=1$: recall from c) $\int u_m^* H_1 u_1 = \frac{W}{2}\delta_{m2}$

$$\Rightarrow \delta E_1^{(2)} = \left(\frac{W}{2}\right)^2\left(\frac{1}{-3E_1^{(0)}}\right) = -\frac{W^2}{12E_1^{(0)}}$$ [1]

$n=2$: note that $H_1 u_2 = W\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a}\cos\frac{\pi x}{a} =$

$$= \frac{W}{2}\sqrt{\frac{2}{a}}\left(\sin\frac{3\pi x}{a} + \sin\frac{\pi x}{a}\right)$$

$$= \frac{W}{2}(u_3 + u_1)$$ [1]

$$\Rightarrow \int u_m^* H_1 u_2 = \frac{W}{2}(\delta_{m3} + \delta_{m1})$$

Thus $$\delta E_2^{(2)} = \left(\frac{W}{2}\right)^2\left[\frac{1}{E_2^{(0)} - E_3^{(0)}} + \frac{1}{E_2^{(0)} - E_1^{(0)}}\right] = \frac{W^2}{4E_1^{(0)}}\left(-\frac{1}{5} + \frac{1}{3}\right)$$ [1]

$$= W^2/(30E_1^{(0)})$$