SEMESTER 2 EXAMINATION 2014-2015

THEORIES OF MATTER, SPACE AND TIME

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answers to Section A and Section B must be in separate answer books

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language word to word® translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

The Lorentz transformation matrix for a boost along the positive x direction by speed v is given by

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Electric and magnetic field components may be written as the elements of the tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} .$$

The metric tensor in our convention is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Throughout the paper we assume the standard notation of c as the speed of light in vacuum, and h as Planck's constant.

Section A

A1. (a) Explain what is meant by an ignorable, or cyclic, coordinate in Lagrangian mechanics and its relationship to a conservation law. [2]

(b) Write down the Lagrangian for a particle of mass m moving in a plane under a central conservative force. For this system identify a cyclic coordinate and the associated conservation law. [2]

A2. In an inertial frame two events occur at the same position but separated by 2 seconds in time. In a second inertial frame, the events are separated by 4 seconds. What is the spatial separation of the events in the second frame? [2]

A3. A pion comes to rest and decays into a muon and a neutrino. By approximating the neutrino mass as $m_\nu \simeq 0$, show that the kinetic energy of the muon is given in $c = 1$ units by $(m_\pi - m_\mu)^2/(2m_\pi)$, where m_π is the pion mass and m_μ is the muon mass. [2]

A4. The two inhomogeneous Maxwell equations can be written in terms of the scalar potential φ and vector potential $\boldsymbol{A}$, using $c = 1$ units, as

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)\varphi - \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A}\right) = \frac{\rho}{\varepsilon_0} \quad ,$$

$$\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2\right)\boldsymbol{A} + \boldsymbol{\nabla}\left(\frac{\partial \varphi}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{A}\right) = \mu_0 \boldsymbol{J} \quad .$$

(a) Write down definitions for the four-gradient ∂^μ, the four-current J^μ, the four-potential A^μ and the field strength tensor $F^{\mu\nu}$. [4]

(b) Show the above equations can be recast in the relativistically covariant form

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad .$$

[2]

TURN OVER

A5. Show that the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\phi = 0$$

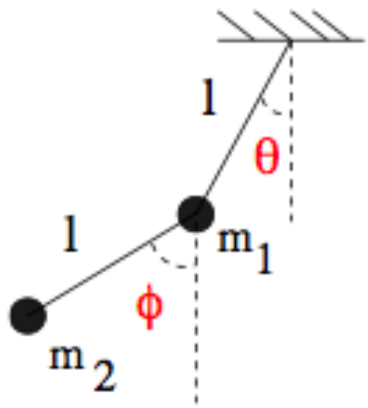
implies that the current

$$j^\mu = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)$$

is conserved, that is, $\partial_\mu j^\mu = 0$. [6]

Section B

B1. (a) A double pendulum consists of a bob of mass m_1 hanging by a light string of length l from a rigid support and a second bob of mass m_2 hanging by an identical string of length l from m_1, as shown in the figure. The pendulum is constrained to swing in a vertical plane. Let the angle between the upper string and the vertical be θ, and that between the lower string and the vertical be ϕ.



Using θ and ϕ as generalised coordinates, write down the Lagrangian for this system. [6]

(b) Suppose $m_1 = m_2 = m$. Show that, for small oscillations about the equilibrium $\theta, \phi \ll 1$, the Euler-Lagrange equations are given by

$$\ddot{\theta} = -\frac{g}{l}\,(2\theta - \phi) \quad , \qquad \ddot{\phi} = -2\,\frac{g}{l}\,(\phi - \theta) \quad .$$

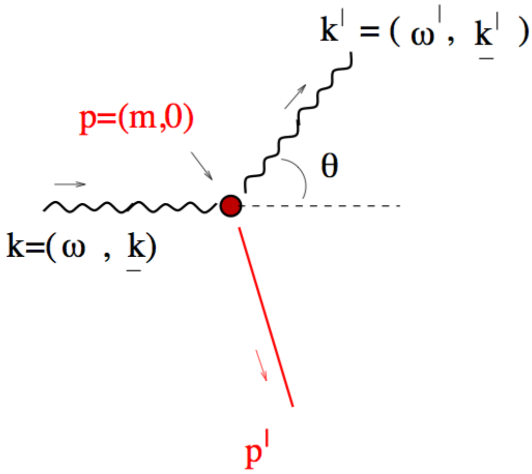
[6]

(c) By solving the Euler-Lagrange equations, determine the normal frequencies of the system and show that the higher frequency is $(\sqrt{2} + 1)$ times the lower frequency. [5]

(d) Show that in the two normal modes the oscillation amplitude of ϕ is $\mp\sqrt{2}$ times that of θ. [3]

TURN OVER

B2. (a) Consider Compton scattering in which a photon with four-momentum $k^\mu = (\omega, \mathbf{k})$, $\omega = |\mathbf{k}|$ (in natural units $\hbar = c = 1$), strikes a particle of mass m at rest and is scattered at angle θ with final four-momentum $k'^\mu = (\omega', \mathbf{k}')$, $\omega' = |\mathbf{k}'|$.



Determine the final photon's energy ω' as a function of ω, m and θ. [6]

(b) A collimated beam of X rays of energy 17.5 keV is incident on a carbon target. Discuss what kinds of scattering processes you expect the X ray beam to undergo, and sketch the wavelength spectrum you expect to be observed at a scattering angle of 90°, including a quantitative indication of the scale. [6]

(c) An electron storage ring contains circulating electrons with energy $E_e = 27$ GeV. Photons with wavelength 514 nm from an argon-ion laser are directed so as to collide head-on with the stored electrons ("inverse Compton scattering"). Show that the maximum scattered photon energy occurs for backscattering ($\theta \sim \pi$) and is approximately given by

$$E'_{\max} \approx \frac{E}{E/E_e + m_e^2/4E_e^2} \, ,$$

where E is the incoming photon energy and m_e is the electron mass. [6]

By numerically evaluating $E'_{\max}$, or otherwise, estimate the maximum factor by which the collision can increase the photon energy. [2]

B3. The Lagrangian for a particle of mass m and charge e in an electromagnetic field may be written as

$$L = \frac{1}{2}m\dot{\boldsymbol{x}}^2 - e\varphi + e\dot{\boldsymbol{x}} \cdot \boldsymbol{A} \ ,$$

where φ is the scalar potential and $\boldsymbol{A}$ is the vector potential.

(a) Show that the Euler-Lagrange equations applied to L give the equation of motion of the particle

$$m\ddot{\boldsymbol{x}} = e(\boldsymbol{E} + \dot{\boldsymbol{x}} \times \boldsymbol{B}) \ ,$$

where $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \dot{\boldsymbol{A}}$, $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. [7]

(b) Evaluate the generalised momenta defined as $p_j = \partial L/\partial \dot{x}_j$ $(j = 1, 2, 3)$. Do they depend on the vector potential? [2]

(c) Define the Hamiltonian H in terms of p_j and L as

$$H = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} - L \ ,$$

and show that it evaluates to

$$H = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2 + e\varphi \ .$$

[6]

(d) Consider the effect of adding a total derivative to the Lagrangian, $L \to L + df/dt$, where f is an arbitrary function, and discuss how this is related with gauge transformations of the potentials. [5]

TURN OVER

B4. A one-dimensional, infinitely deep square-well potential, with $V(x) = 0$ for $0 < x < a$ and $V(x) \to \infty$ everywhere else, is perturbed by the hamiltonian

$$H_1 = W \cos\left(\frac{\pi x}{a}\right)$$

for $0 < x < a$, where W is constant ($H_1 = 0$ elsewhere). The unperturbed energy eigenvalues and eigenfunctions of a particle of mass m are known and given by

$$E_n^{(0)} = \frac{\hbar^2 \pi^2 n^2}{2ma^2} \quad , \quad u_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad .$$

(a) Sketch the perturbed potential well as a function of x. [2]

(b) Show that to first order of perturbation theory the energy levels of the particle are unchanged by H_1. [4]

(c) Find the first-order correction to the ground-state wave function. [5]

In which region of x is the particle more localised as an effect of the perturbation? [3]

What is the condition on W for perturbation theory to give a sensible approximation to the wave function? [2]

(d) Show that the second-order energy shifts to the first two energy levels $n = 1$ and $n = 2$ are given by

$$\delta E_1^{(2)} = -\frac{W^2}{12 E_1^{(0)}} \quad , \quad \delta E_2^{(2)} = \frac{W^2}{30 E_1^{(0)}} \quad ,$$

where $E_1^{(0)} = \hbar^2 \pi^2 / (2ma^2)$. [4]

END OF PAPER