SEMESTER 2 EXAMINATION 2012-2013

THEORIES OF MATTER, SPACE AND TIME

Duration: 120 MINS (2 hours)

This paper contains 10 questions.

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Copyright 2013 © University of Southampton

The Lorentz transformation matrix for a boost along the positive x direction by speed v is given by

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Electric and magnetic field components may be written as the elements of the tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}.$$

The metric tensor in our convention is

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Copyright 2013 © University of Southampton

Section A

A1. State Fermat's Principle for the propagation of light. [1]

A2. Express the generalised momentum associated with a single generalised co-ordinate q in terms of the Lagrangian $L(q, \dot{q}, t)$. State the conditions on the Lagrangian such that the generalised momentum and the Hamiltonian are conserved. [3]

A3. What are the two postulates of Special Relativity? [2]

A4. An ultra-relativistic cosmic-ray antiproton with energy E_p collides with a static proton in the upper atmosphere. Using four-momenta, show that the maximum mass of a particle created by the collision is given by approximately,

$$M_{\text{max}} \approx \sqrt{\frac{2E_p m_p}{c^2}}.$$

[3]

A5. Starting with the relativistic energy-momentum relation

$$E^2 = m^2c^4 + \vec{p}^2c^2,$$

where m is the mass of the particle, $\vec{p}$ is its momentum and E its energy, write down the energy and momentum as differential operators and obtain the Klein-Gordon equation involving the $\Box$ operator and the wavefunction ϕ. [4]

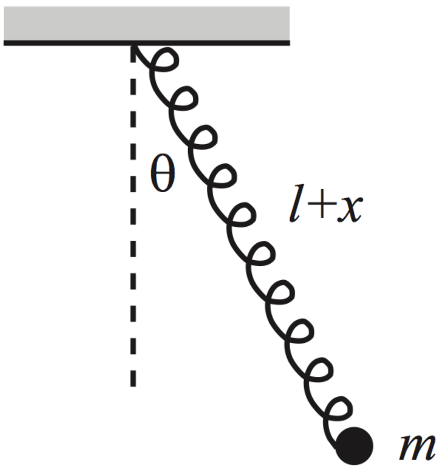
A6. Given the tensor $F^{\mu\nu}$, use the metric tensor (twice) to determine all the components of the tensor $F_{\mu\nu}$ for $\mu = 0$ and all ν values. Then find F_{12}, F_{13} and F_{23}. Finally, using symmetry arguments, write down the entire tensor $F_{\mu\nu}$. [7]

TURN OVER

Copyright 2013 © University of Southampton

Section B

B1. (a) Consider a pendulum made from a massless spring with a mass m on the end. The spring is arranged to lie in a straight line (which can be done by wrapping the spring around a rigid light rod). The equilibrium length of the spring is l. At some time t the spring has length $l + x$ and its angle with the vertical is θ, as depicted in the figure below.



Assuming that the motion takes place in a vertical plane, show that the Euler-Lagrange equations of motion for x and θ are:

$$m\ddot{x} = m(l + x)\dot{\theta}^2 + mg\cos\theta - kx,$$

$$m(l + x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta,$$

where k is the spring constant and g is the acceleration due to gravity. State the physical interpretation of each of the terms $m(l + x)\dot{\theta}^2$ and $2m\dot{x}\dot{\theta}$.

[12]

(b) Write down an approximate form of the equations of motion in a small angle approximation. Rewrite the equations using $\Delta x = x - (mg/k)$. [4]

At time $t = 0$ the spring is extended by an amount $\Delta x = x_0$ and is released from rest at a small angle θ_0. Find the approximate solution for $\Delta x(t)$. Hence find the approximate equation satisfied by $\theta(t)$ in this case. [4]

Copyright 2013 © University of Southampton

B2. (a) The components of four momentum in terms of energy and momentum are

$$p^\mu = (E/c, \vec{p}).$$

By explicitly transforming the components of p^μ show that $p^\mu p_\mu$ is invariant under a Lorentz transformation with speed v in the positive x direction. [6]

(b) Four momentum can also be defined as

$$p^\mu = m\frac{dx^\mu}{d\tau}$$

where x^μ is the position four vector of a particle of mass m and τ is the proper time. Using this definition, evaluate $p^\mu p_\mu$ in the rest frame of the particle and compare the result to that obtained using the previous definition. [4]

(c) In Compton Scattering a high energy photon is scattered from a static free electron in a metal. Show, using four momentum conservation, that the change in wavelength, $\Delta\lambda$, of the photon depends on its scattering angle, θ, according to

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\theta).$$

[10]

TURN OVER

Copyright 2013 © University of Southampton

B3. (a) The laws of electromagnetism can be summarised as,

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu, \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

where $J^\nu = (\rho c, \vec{J})$ and $A^\nu = (\phi/c, \vec{A})$.

State in words the meaning of each of the components ρ, $\vec{J}$, ϕ and $\vec{A}$. [2]

Applying the first equation for $(\nu = 0)$ and $(\nu = 1)$, determine two Maxwell equations for the specified components. Then applying the second equation for $(\mu = 0,\ \nu = 1)$ and $(\mu = 1,\ \nu = 2)$, determine two further equations for the specified components. [8]

Hence write the four equations determined above in three-vector form. [2]

(b) By combining the first two equations given in part (a), show that in Lorentz gauge,

$$\Box A^\nu = \mu_0 J^\nu.$$

[2]

Hence show that in the vacuum of free space there are plane wave solutions propagating in the x direction with wave vector k and angular frequency ω, of the form

$$A^\nu = A_0^\nu e^{i(kx-\omega t)},$$

providing ω and k satisfy a particular condition. [3]

If $\vec{A}_0$ is in the $\hat{z}$ direction, calculate the magnetic field $\vec{B}$ for this solution. [3]

Copyright 2013 © University of Southampton

B4. Consider solutions of an unperturbed one dimensional time independent Schrodinger equation (with i an integer labelling the solution)

$$H_0\phi_i(x) = E_{0i}\phi_i(x).$$

(a) Explain with words and equations what it means for the set of eigenfunctions $\phi_i(x)$ to be both complete and orthogonal. [4]

(b) If a small perturbing contribution to the Hamiltonian, H_p, is introduced, the eigenfunction undergoes a shift $\phi_i \to \psi_i = \phi_i + \delta\phi_i$ and its energy is shifted by δE_i. Show that

$$\phi_j^* H_0 \delta\phi_i + \phi_j^* H_p \phi_i = \phi_j^* E_{0i} \delta\phi_i + \phi_j^* \delta E_i \phi_i.$$

[4]

(c) Express $\delta\phi_i$ as a complete set of states and then integrate the expression given in part (b) over all x to show

$$\delta E_i = \int \phi_i^* H_p \phi_i dx.$$

[5]

Derive an expression for the perturbation to the ground state wavefunction.

[3]

(d) The ground state wave function of the Schrodinger equation with a simple harmonic potential is given by

$$\phi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}$$

where α is a constant. What is the shift in the energy of the ground state if a perturbation to the potential $\Delta V = \lambda x^6$ is introduced? [4]

Note the integral result:

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\alpha x^2} x^{2n} dx = (-1)^n \frac{d^n}{d\alpha^n}\left(\sqrt{\frac{\pi}{\alpha}}\right).$$

END OF PAPER

Copyright 2013 © University of Southampton