SEMESTER 2 EXAMINATION 2012-2013

STELLAR EVOLUTION

Duration: 120 MINS (2 hours)

This paper contains 8 questions.

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

**Section A** carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

**Section B** carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

# Formula Sheet

Stellar structure equations:

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dl}{dm} = \epsilon_{\rm nuc}$$

$$\frac{dT}{dm} = -\frac{Gm}{4\pi r^4}\frac{T}{P}\nabla \quad \text{with} \quad \nabla = \begin{cases} \nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} & \text{if} \quad \nabla_{\rm rad} \le \nabla_{\rm ad} \\ \nabla_{\rm ad} & \text{if} \quad \nabla_{\rm rad} > \nabla_{\rm ad} \end{cases}$$

Eddington luminosity:

$$L_{\rm Edd} = \frac{4\pi cGM}{\kappa} = 3.8 \times 10^4 \left(\frac{M}{M_\odot}\right)\left(\frac{0.34\,{\rm cm}^2/{\rm g}}{\kappa}\right) L_\odot$$

Diffusion equations:

$$F = -D\nabla U = -K\nabla T \quad \text{with} \quad K = D\,C_V = \frac{1}{3} v l_{\rm fp} C_V$$

Mean free path:

$$l_{\rm fp} = \frac{1}{\kappa\rho} = \frac{1}{n\sigma}$$

Ideal gas equation of state:

$$P_{\rm gas} = nkT = \frac{R}{\mu}\rho T = \frac{k}{\mu m_u}\rho T$$

Fermi momentum:

$$p_F = h\left(\frac{3}{8\pi}n_e\right)^{1/3}$$

Fermion equation of state (non-relativistic):

$$P_{\mathrm{e,NR}} = K_{\mathrm{NR}} \left(\frac{\rho}{\mu_e}\right)^{5/3} \quad \text{with} \quad K_{\mathrm{NR}} = \frac{h^2}{20 m_e m_u^{5/3}} \left(\frac{3}{\pi}\right)^{2/3} = 1.0036 \times 10^{13} \quad [\mathrm{cgs}]$$

Fermion equation of state (extremely relativistic):

$$P_{\mathrm{e,ER}} = K_{\mathrm{ER}} \left(\frac{\rho}{\mu_e}\right)^{4/3} \quad \text{with} \quad K_{\mathrm{ER}} = \frac{hc}{8 m_u^{4/3}} \left(\frac{3}{\pi}\right)^{1/3} = 1.2435 \times 10^{15} \quad [\mathrm{cgs}]$$

Boson equation of state:

$$P_\gamma = \frac{1}{3} a T^4$$

Mass-luminosity relationship for main sequence stars:

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^3$$

Lane-Emden equation for a polytrope $P = K\rho^\gamma = K\rho^{(n+1)/n}$:

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2 \frac{dw}{dz}\right) + w^n = 0$$

with $\rho = \rho_c w^n$
and $r = \alpha z$, where $\alpha = \sqrt{\frac{n+1}{4\pi G} K \rho_c^{(1-n)/n}}$.

Physical properties of the Lane-Emden equations:

$$R = \alpha z_n$$

$$M = 4\pi\alpha^3 \rho_c \Theta_n$$

$$K = N_n G M^{(n-1)/n} R^{(3-n)/n} \quad \text{with} \quad N_n = \frac{(4\pi)^{1/n}}{n+1} \Theta_n^{(1-n)/n} z_n^{(n-3)/n}$$

$$P_c = W_n \frac{GM^2}{R^4}$$

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Numerical values for polytropic models with index n:

| $n$ | $z_n$ | $\Theta_n$ | $\rho_c/\rho$ | $N_n$ | $W_n$ |
|---|---|---|---|---|---|
| 0 | 2.44949 | 4.89898 | 1.00000 | ... | 0.119366 |
| 1 | 3.14159 | 3.14159 | 3.28987 | 0.63662 | 0.392699 |
| 1.5 | 3.65375 | 2.71406 | 5.99071 | 0.42422 | 0.770140 |
| 2 | 4.35287 | 2.41105 | 11.40254 | 0.36475 | 1.638183 |
| 3 | 6.89685 | 2.01824 | 54.1825 | 0.36394 | 11.05068 |
| 4 | 14.97155 | 1.79723 | 622.408 | 0.47720 | 247.559 |
| 4.5 | 31.8365 | 1.73780 | 6189.47 | 0.65798 | 4921.84 |
| 5 | $\infty$ | 1.73205 | $\infty$ | $\infty$ | $\infty$ |

Physical constants:

$c = 3 \times 10^{10}$ cm/s

$k = 1.38 \times 10^{-16}$ erg/K

$R = 8.31 \times 10^{7}$ erg/K/mol

$m_u = 1.66 \times 10^{-24}$ g

$m_e = 9.11 \times 10^{-28}$ g

$h = 6.62 \times 10^{-27}$ erg s

$G = 6.67 \times 10^{-8}$ cm$^3$/g/s$^2$

$1eV = 1.60 \times 10^{-12}$ erg

$a = 7.56 \times 10^{-15}$ erg/cm$^3$/K$^4$

$\sigma_{SB} = 5.67 \times 10^{-5}$ erg/cm$^2$/K$^4$/s

$\mathrm{AU} = 1.496 \times 10^{13}$ cm

$M_\odot = 1.99 \times 10^{33}$ g

$R_\odot = 6.96 \times 10^{10}$ cm

$L_\odot = 3.9 \times 10^{33}$ erg/s

$T_\odot = 5780$ K

## Section A

**A1.** Explain why main sequence stars stars have a maximum mass limit. Derive an approximate value for it. You can assume that the stellar opacity is entirely due to electron scattering, i.e. $\kappa = 0.34\,\mathrm{cm}^2\,\mathrm{g}^{-1}$. [ 4 ]

**A2.** The figure below shows the Rosseland mean opacity as a function of temperature and density. Darker grey shades indicate larger opacities. The stellar structures of 1, 10 and 100 $M_\odot$ stars are drawn with thick black lines. Particular regions of interest are marked with numbers inside squares on this diagram. Please indicate:

(a) The phenomenon responsible for the drastic decrease of opacity in region #1. [ 1 ]

(b) One of the two main sources of opacity occurring in region #2. [ 1 ]

(c) One of the two main sources of opacity occurring in region #3. [ 1 ]

(d) The main source of opacity occurring in region #4. [ 1 ]

(e) The main source of opacity occurring in region #5. [ 1 ]

**A3.** Complete the following nuclear reactions drawn from the first half of the CNO cycle. Note that there might be more than one particle missing in place of the "?". [ 5 ]

$$
\begin{aligned}
{}^{12}C + {}^{1}H &\rightarrow {}^{13}N + ? \\
{}^{13}N &\rightarrow {}^{13}C + ? \\
{}^{13}C + {}^{1}H &\rightarrow {}^{14}N + ? \\
{}^{14}N + ? &\rightarrow {}^{15}O + ? \\
{}^{15}O &\rightarrow {}^{15}N + ?
\end{aligned}
$$

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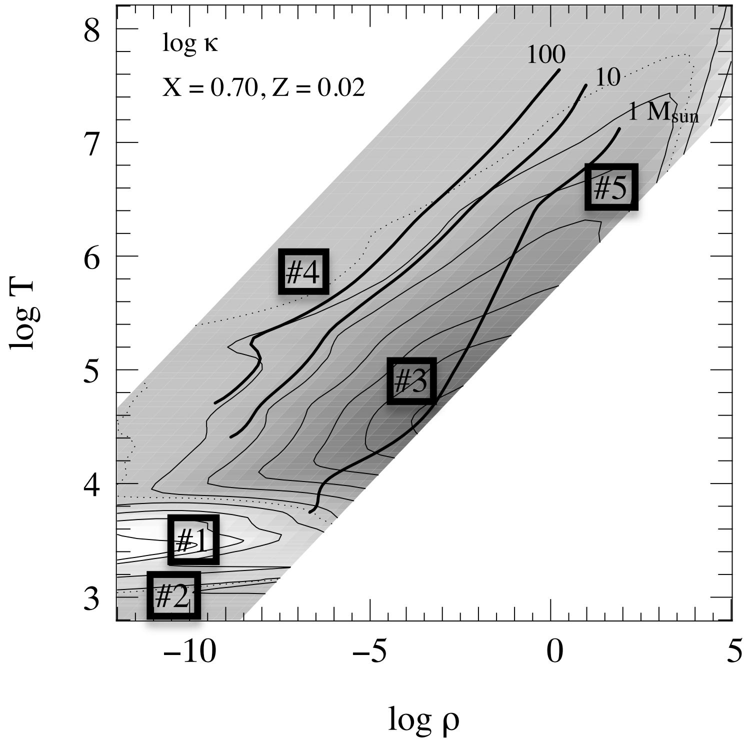


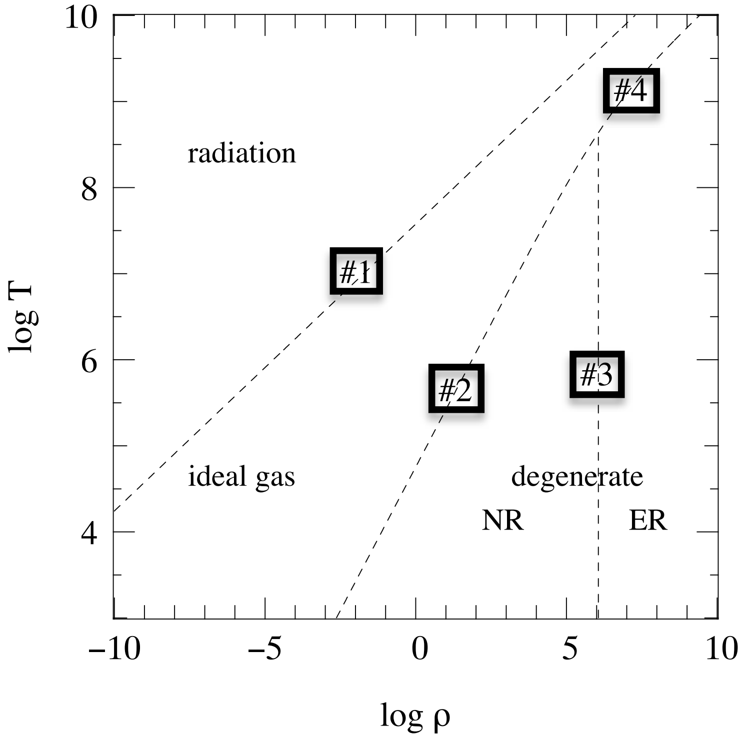

$$^{15}N + {}^{1}H \rightarrow {}^{12}C + ?$$
$$^{15}N + {}^{1}H \rightarrow {}^{16}O + ?$$

**A4.** Explain two major effects played by convection in the evolution of stars. [ 3 ]

**A5.** Describe what role neutrinos play during (a) the main sequence phase, (b) the red giant and the white dwarf evolution, and (c) the core-collapse supernova? [ 3 ]

## Section B

**B1.** (a) The figure below displays the regions in temperature-density plane described by various equations of state. Find the equations describing the boundaries between these regions, each of which is marked by a number inside a square. You should express the relationship between $T$ and $\rho$ as a numerical prefactor times a $\mu$ and $\mu_e$. In the figure, 'NR' and 'ER' refer to non-relativistic and extremely relativistic, respectively. [ 12 ]




(b) The Chandrasekhar mass provides the upper mass limit of white dwarfs at which a Fermion gas reaches the relativistic limit. Show that this limit can be expressed as: [ 6 ]

$$M_{\rm Ch} = 5.836\,\mu_e^{-2}\,M_\odot\,.$$

(c) Name another effect, besides chemical composition, that can affect the Chandrasekhar mass and briefly explain how it acts physically. [ 2 ]

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**B2.** Except for a handful of cases, solving the stellar structure equations requires performing numerical integrations with appropriate boundary conditions. The homology relations allow one to determine the overall properties of stars of different masses, radii, temperatures and luminosities having similar equations of state.

(a) Using the following dimensionless parameterisation of the mass:

$$x = \frac{m_1}{M_1} = \frac{m_2}{M_2}\,,$$

and the following homology identity for the radius:

$$\frac{r_1(x)}{R_1} = \frac{r_2(x)}{R_2}\,,$$

along with the first equation of stellar structure for mass conservation, show that there exists a very simple relationship between density, radius and mass. What is the implication for (i) the density in a given mass shell, (ii) the average density and (iii) the central density? [ 8 ]

(b) Use the same mass-radius parameterisation, along with the second equation of stellar structure for hydrostatic equilibrium, to find the relationship linking pressure, mass and radius. What is the implication for (i) the pressure in a given mass shell and (ii) the central pressure? [ 8 ]

(c) Combine the relationship that you have obtained in (a) with the one you have obtained in (b) in order to derive a relationship between pressure and density in stars having a similar equation of state. [ 4 ]

**B3.** Convection is a very efficient transport mechanism in stars.

(a) In its simplest form, this criterion for stability against convection relates the rate of change of density with respect to pressure in a star, $d\log\rho/d\log P$, to that of an adiabatic change, $(d\log\rho/d\log P)_{\rm ad} \equiv \gamma_{\rm ad}^{-1}$. Derive the condition for stability against convection.

In order to do so, consider of blob of matter having a certain density and pressure that are equal to the ambient stellar density and pressure. Imagine that this blob is transported upward fast enough such that it experiences an adiabatic change to adapt to the new ambient conditions while its surroundings remain in hydrostatic equilibrium.

i) Sketch a diagram of the physical situation presented above. [ 2 ]

ii) On a $\log\rho - \log P$ plot, display the physical situation presented above. This involves drawing the curve for an adiabatic change as well as one for the unstable and one for the stable situation. [ 3 ]

iii) Derive the equation for stability against convection. You are not required to make a full mathematical derivation; a pseudo-derivation based on arguments drawn from the diagram and the plot you made are sufficient. [ 5 ]

(b) The Schwarzschild criterion expresses the convection stability in a purely radiative, chemically homogeneous star. It relates the radiation pressure gradient to the adiabatic pressure gradient as follows:

$$\nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa l}{m} > \nabla_{\rm ad}\,.$$

Name two possible conditions that occur in a star that will lead to a violation of the Schwarzschild criterion. For each of them, specify the type of star/situation in which this condition would happen. [ 6 ]

(c) Describe the thin shell instability. What evolution stage is it mostly relevant for? What kind of stellar mass does it pertain to? [ 4 ]

**END OF PAPER**