Synoptic paper 2015

Phys 3017/6015

Model answers

A1

Dispersion: wave speed varies with frequency (or with wave length, $\frac{dv}{d\lambda} \neq 0$)

$\omega = \omega(k)$ specifies dispersion relation [2]

Phase velocity: $v = \omega / k$ [1]

Group velocity: $g = \frac{d\omega}{dk}$ [1]

A2

$$d \sin\vartheta_1 = n\lambda_1 \quad , \quad d \sin\vartheta_2 = n\lambda_2$$

where $d = 300^{-1}\ 10^{-3}\ m = 3.3\ 10^{-6} m$ (1)

For $n=1$, $\Delta\vartheta = \arcsin\frac{\lambda_2}{d} - \arcsin\frac{\lambda_1}{d} = 0.01°$ (1)

Highest order $\Rightarrow \sin\vartheta = 1 \Rightarrow n_{max} = [\frac{d}{\lambda}] = [5.7] = 5$ (1)

For $n=5$, $\Delta\vartheta = \arcsin\frac{5\lambda_2}{d} - \arcsin\frac{5\lambda_1}{d} = 0.11°$ (1)

A3

Commutator $\qquad\qquad \frac{h}{2\pi} \leftarrow$ Planck constant

$$[J_x, J_y] \equiv J_x J_y - J_y J_x = i\hbar J_z$$

The commutator of any two components of angular momentum is $(i\hbar) \times$ (third component) [1] $\qquad [J_i, J_j] = i\varepsilon_{ijk} J_k$

$[J_x, J_y] \neq 0 \Rightarrow$ there is in general no complete set of states for which both J_x and J_y have definite values [1]

On the other hand $[J^2, J_i] = 0$ for any i [1]

$\Rightarrow$ can find complete set of states for which both J^2 and one component of J have well-defined values. [1]

A4

2-particle wave function $\psi(1,2)$

1st particle in state 1 ; 2nd particle in state 2

Indistinguishable particles $\Rightarrow$ exchange symmetry:

$$|\psi(2,1)|^2 = |\psi(1,2)|^2$$ [1]

i.e. $\psi(2,1) = e^{i\phi}\psi(1,2)$

Swap twice $\Rightarrow$ $\psi(2,1) = e^{2i\phi}\psi(2,1)$

i.e. $e^{2i\phi} = 1 \Rightarrow e^{i\phi} = \pm 1$ [1]

That is,

either $\psi(2,1) = \psi(1,2)$ bosons → integer spin

or $\psi(2,1) = -\psi(1,2)$ fermions → half-integer spin [1]

Pauli exclusion principle for fermions:

if state 1 = state 2 : $\psi(1,1) = -\psi(1,1) = 0$ [1]

two fermions cannot occupy same state

A5

Ionisation energy $\sim R = 13.6\ \text{eV}$ [1]

Temperature of ionisation

$$k_B T \sim R$$ [1]

$$\Downarrow$$

$$T \sim \frac{R}{k_B} \sim \frac{13.6\ 1.6\ 10^{-19}}{1.38\ 10^{-23}}\ \text{K} \sim 1.5\ 10^{5}\ \text{K}$$ [2]

B1

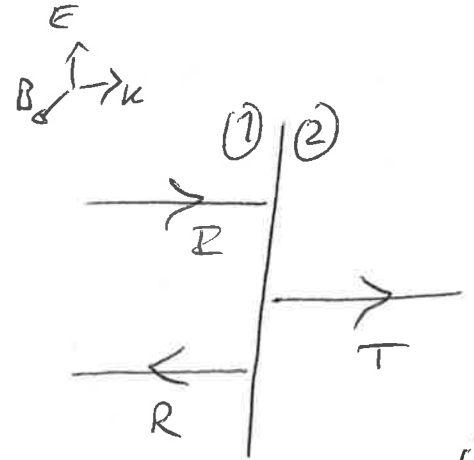
a) $\rho = 0, \quad \underline{J} = 0 \quad \Rightarrow \quad \underline{\nabla} \cdot \underline{E} = 0 \qquad \underline{\nabla} \cdot \underline{B} = 0$ [4]

$$\underline{\nabla} \wedge \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \underline{\nabla} \wedge \underline{B} = \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$$

b) $$\underbrace{\underline{\nabla} \wedge (\underline{\nabla} \wedge \underline{E})}_{\nabla \underbrace{\underline{\nabla} \cdot \underline{E}}_{=0} - \underline{\nabla}^2 \underline{E}} = -\frac{\partial}{\partial t} \underbrace{\underline{\nabla} \wedge \underline{B}}_{\mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}} \Rightarrow -\underline{\nabla}^2 \underline{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2}$$ [2]

that is, $\nabla^2 \underline{E} - \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2} = 0$

wave equation with wave speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ [2]



c) continuity of $\perp$ components of the fields is trivially satisfied; continuity of // components:

$E_1^{//} = E_2^{//} \Rightarrow E_I + E_R = E_T$ [2]

$H_1^{//} = H_2^{//} \Rightarrow \frac{1}{v_1} E_I - \frac{1}{v_1} E_R = \frac{1}{v_2} E_T$

$\mu_1 \simeq \mu_2 \simeq \mu_0$

i.e., $(v_j \simeq 1/\sqrt{\mu_0 \varepsilon_j})$ $\quad \sqrt{\varepsilon_1}\, E_I - \sqrt{\varepsilon_1}\, E_R = \sqrt{\varepsilon_2}\, E_T$ [2]

Solve for E_R and E_T: $\quad E_I + E_R = E_T = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} (E_I - E_R)$

$$\Rightarrow \quad E_R = E_I \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \qquad [2]$$

$$E_T = E_I + E_R = E_I \left(1 - \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}\right) = E_I \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \qquad [2]$$

Power per unit area: $S = \frac{1}{\mu_0} E \frac{E}{v} = \sqrt{\frac{\varepsilon}{\mu_0}} E^2$ [1]

$$\Rightarrow \quad S_I = \sqrt{\frac{\varepsilon_1}{\mu_0}} E_I^2 \; ; \; S_R = \sqrt{\frac{\varepsilon_1}{\mu_0}} E_R^2 \; ; \; S_T = \sqrt{\frac{\varepsilon_2}{\mu_0}} E_T^2 \qquad [1]$$

Energy conservation $\Rightarrow$ $S_I = S_R + S_T$ i.e. $1 = \frac{S_R}{S_I} + \frac{S_T}{S_I}$ [1]

verify: $\frac{S_R}{S_I} + \frac{S_T}{S_I} = \left(\frac{E_R}{E_I}\right)^2 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \left(\frac{E_T}{E_I}\right)^2 =$

inserting expressions at the top we have

$$= \left(\frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}\right)^2 + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \left(\frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}\right)^2 =$$

$$= \frac{(\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})^2}{(\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})^2} = 1 \qquad [1]$$

B2

a) $H\psi = E\psi$ TISE (1)

$i\hbar \frac{\partial \psi}{\partial t} = H\psi$ TDSE (1)

The wave function of any system must satisfy the TDSE. [1]

The TISE is satisfied if the system is in a state of well-defined energy (stationary state) [1]

The wave function contains a complete set of information about the system, so it must contain complete initial conditions for solution of the time evolution equation. [1]

This would not be possible if the TDSE were second order in t. [1]

b) $E_n = \varepsilon, 4\varepsilon, 9\varepsilon, 16\varepsilon, 25\varepsilon, \ldots$

normalisation of $\psi(0)$: $\|\psi(0)\|^2 = 0.2^2 + 0.3^2 + 0.4^2 + 0.843^2 = 1$

$P(E < 6\varepsilon) = P(\varepsilon \text{ or } 4\varepsilon) = 0.2^2 + 0.3^2 = 0.13$ [3]

Energy expectation value

$$\langle E\rangle = 0.2^2\,\varepsilon + 0.3^2\,4\varepsilon + 0.4^2\,9\varepsilon + 0.843^2\,16\varepsilon =$$

$$= 13.2\varepsilon \qquad [2]$$

rms deviation

$$\langle E^2\rangle = 0.2^2\,\varepsilon^2 + 0.3^2\,16\varepsilon^2 + 0.4^2\,81\varepsilon^2 + 0.843^2\,256\varepsilon^2 =$$

$$= 196.2\,\varepsilon^2$$

$$\Rightarrow \Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{196.2\,\varepsilon^2 - 13.2^2\,\varepsilon^2}$$

$$= 4.69\,\varepsilon \qquad [2]$$

c)

$$\psi(t) = 0.2\,e^{-\frac{i}{\hbar}\varepsilon t}\,u_1 + 0.3\,e^{-\frac{i}{\hbar}4\varepsilon t}\,u_2 + 0.4\,e^{-\frac{i}{\hbar}9\varepsilon t}\,u_3 + 0.843\,e^{-\frac{i}{\hbar}16\varepsilon t}\,u_4 \qquad [4]$$

d) after measurement $\psi = u_4$ [2]

if energy is measured again, $E = 16\varepsilon$ with certainty (1)

B3

Density of modes:

2 photon polarization states

a) $$g(\omega)\, d\omega = 2\, g(k)\, dk = 2 \frac{4\pi k^2}{(2\pi)^3} dk \qquad [2]$$

$$\omega = ck \Rightarrow \quad g(\omega) = \frac{\omega^2}{\pi^2 c^3} \qquad [2]$$

Energy density: $$u = \int_0^\infty \hbar\omega \frac{1}{e^{\beta\hbar\omega}-1} \frac{\omega^2}{\pi^2 c^3} d\omega = \qquad [2]$$

$$\omega \to x = \beta\hbar\omega$$

$$= \frac{\hbar}{\pi^2 c^3} \frac{1}{(\hbar\beta)^4} \underbrace{\int_0^\infty \frac{x^3}{e^x - 1} dx}_{\pi^4/15} = \frac{\pi^2 k_B^4 T^4}{15 \hbar^3 c^3} \qquad [2]$$

Thus $$\Phi = \frac{uc}{4} = \sigma T^4 \Rightarrow \sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} \qquad [2]$$

b) Total power radiated from Sun's surface =

$$= \sigma T_s^4\, 4\pi R_s^2 \qquad [2]$$

Isotropic emission & satellite absorbing as a black-body disk of radius r

$$\Rightarrow \text{Power absorbed} = \frac{\sigma T_s^4 R_s^2}{R^2} \pi r^2 \qquad [2]$$

orbit radius

In equilibrium, power absorbed = power radiated

$$\Rightarrow \quad \frac{\sigma T_s^4 R_s^2 \pi r^2}{R^2} = \sigma T^4 4\pi r^2 \qquad [2]$$

Therefore $T = T_s \left(\frac{R_s}{2R}\right)^{1/2}$

$$\left.\begin{array}{l} R_s = 6.955\ 10^8\ \mathrm{m} \\ R = 1.496\ 10^{11}\ \mathrm{m} \\ T_s = 5700\ \mathrm{K} \end{array}\right\} = 274.8\ \mathrm{K} \qquad [2]$$

c) With power supply on, equilibrium relation is modified to

$$W + \frac{\sigma T_s^4 R_s^2 \pi r^2}{R^2} = \sigma T^4 4\pi r^2 \qquad [1]$$

So

$$T = \left(\frac{W + \frac{\sigma T_s^4 R_s^2 4\pi r^2}{R^2}}{4\pi r^2 \sigma} \right)^{1/4}$$

$$= 276.5\ \mathrm{K} \qquad [1]$$

new equilibrium temperature higher by 1.7 K

B4

a) $$E = T + V = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\vartheta}^2 - \frac{G M_* m}{r} \qquad [1]$$

Because the gravitational force is central, angular momentum is conserved: $J = m r^2 \dot{\vartheta} = \text{constant}$ [1]

$$\Rightarrow E = \underbrace{\frac{1}{2} m \dot{r}^2}_{\text{radial KE}} + \underbrace{\underbrace{\frac{J^2}{2 m r^2}}_{\text{tangential KE}} - \underbrace{\frac{G M_* m}{r}}_{\text{gravitational PE}}}_{\text{effective potential } U(r)} \qquad [1]$$

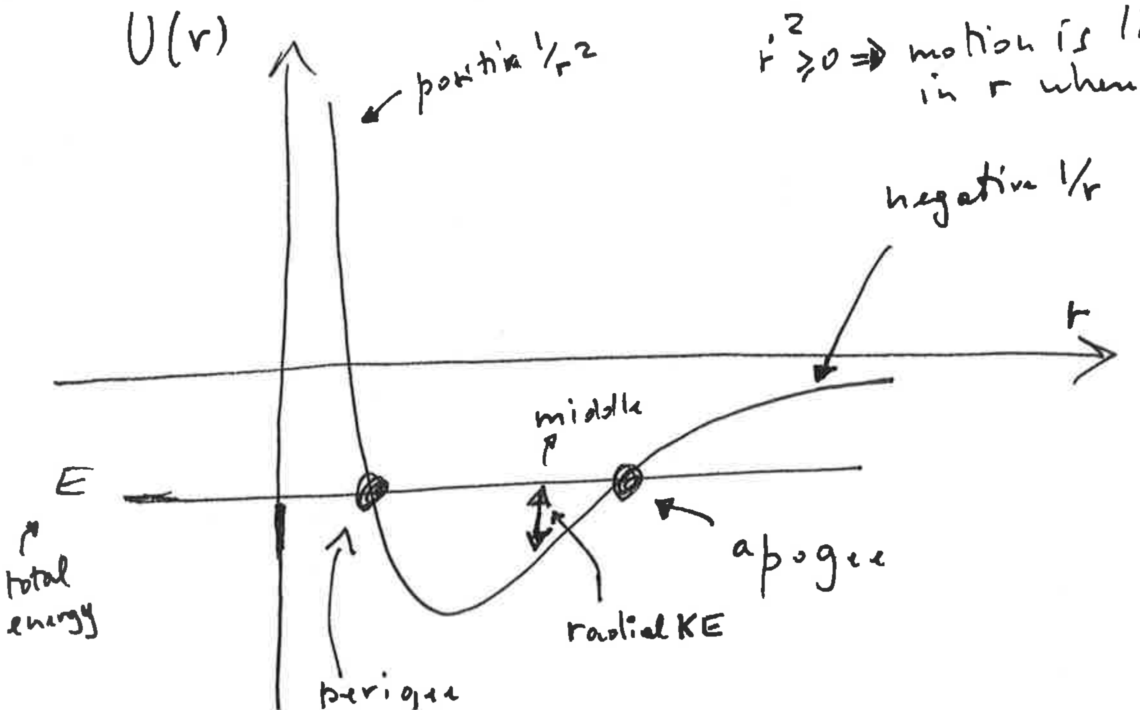
"radial energy equation"

b) For given J the energy equation has the form one would get for a particle moving in one dimension in an effective potential [1]

$$U(r) = \underbrace{\frac{J^2}{2 m r^2}}_{\text{centrifugal}} - \underbrace{\frac{G M_* m}{r}}_{\text{gravitational}}$$

$$-\frac{\partial}{\partial r}\left(\frac{J^2}{2 m r^2}\right) = \frac{J^2}{m r^3} = m r \dot{\vartheta}^2 = \text{centrifugal force} \qquad [1]$$

$\dot{r}^2 \geq 0 \Rightarrow$ motion is limited to range in r where $U(r) \leq E$



[2]

[3]

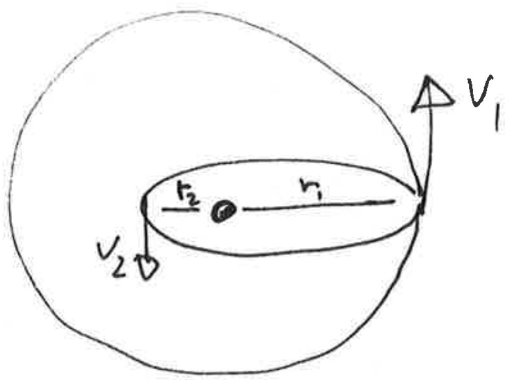
c) circular orbit: $\frac{mv^2}{r} = \frac{GM_e m}{r^2} \Rightarrow v = \sqrt{\frac{GM_e}{r}}$ [2]

$$T = \frac{2\pi r}{v} = \frac{2\pi r^{3/2}}{\sqrt{GM_e}} \Rightarrow r = (GM_e)^{1/3} \left(\frac{T}{2\pi}\right)^{2/3}$$ [2]

$T = 24h \Rightarrow \simeq 4.2 \cdot 10^7 m$ [1]

Thus $v = \sqrt{\frac{GM_e}{r}} \simeq 3080 \ m/s$ [1]

d)



Conservation of $J \Rightarrow m v_2 r_2 = m v_1 r_1$ [1]

Conservation of $E \Rightarrow \frac{1}{2} m v_1^2 - \frac{GM_e m}{r_1} = \frac{1}{2} m v_2^2 - \frac{GM_e m}{r_2}$ [1]

Then $\frac{1}{2} m v_1^2 \left[1 - \left(\frac{r_1}{r_2}\right)^2\right] = GM_e m \left(\frac{1}{r_1} - \frac{1}{r_2}\right)$

$$\Rightarrow v_1 = \sqrt{\frac{2GM_e r_2}{r_1(r_1+r_2)}} \simeq 1816 \ m/s$$ [1]

So the change in velocity needed is

$v - v_1 \simeq 1260 \ m/s$ [1]

C1

a) The spin orbit coupling comes from the interaction of the electron's magnetic moment $\underline{\mu} = -g\frac{e}{2m}\underline{S}$ with the magnetic field seen in the rest frame of the electron as they moves through the Coulomb field of the nuclear charge: $H = -\underline{\mu} \cdot \underline{B}$. The magnetic field is proportional to the rate at which the electron orbits and thus to orbital angular momentum. So the interaction has the form $-\underline{\mu} \cdot \underline{B} \propto \underline{S} \cdot \underline{L}$ [2] [2]

As a result of spin orbit coupling m_ℓ and m_s are no longer good quantum numbers but the total angular momentum quantum numbers j and m_j are. [1] [1]

So a valid set is n, ℓ, j, m_j. [1]

b) $\underline{\mu} = -g\frac{e}{2m}\underline{S}$, $g \simeq 2$ [1]

$$\Rightarrow \underline{B} \simeq -\frac{1}{mc^2}\frac{1}{r}\frac{\partial \phi}{\partial r}\underline{L}$$ [1]

$$= -\frac{1}{mc^2}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{Ze}{4\pi\varepsilon_0 r}\right)\underline{L}$$ [1]

$$= \frac{1}{mc^2}\frac{Ze}{4\pi\varepsilon_0 r^3}\underline{L}$$ [1]

c) H (2p) $\Rightarrow$ $Z = 1$, $n = 2, \ell = 1$ [1]

Last, $\langle \frac{1}{r^3} \rangle = \left(\frac{Z}{n a_0}\right)^3 \frac{2}{\ell(\ell+1)(2\ell+1)}$

$$\Rightarrow B \simeq \underbrace{\frac{e\hbar}{2m}}_{\mu_B} \underbrace{\frac{1}{4\pi\varepsilon_0 c^2}}_{\frac{\mu_0}{4\pi}} \frac{1}{a_0^3} \underbrace{\frac{Z^4 4}{n^3 l(l+1)(2l+1)}}_{1/12} \qquad [1]$$

$$\simeq \frac{9\,10^{-24}\;\;10^{-7}}{(5\,10^{-11})^3}\frac{1}{12} \simeq 0.6\,T \qquad [1]$$

$$\left\langle -\frac{1}{r}\frac{\partial\phi}{\partial r}\right\rangle = \frac{Ze}{4\pi\varepsilon_0}\underbrace{\left\langle\frac{1}{r^3}\right\rangle}_{\propto Z^3} \Rightarrow \text{fine-structure} \sim Z^4 \qquad [3]$$

d) ΔE_{nj} degenerate with respect to l and m_j [1]

For given j, $l = j - 1/2$ or $j + 1/2$, except for the highest $j = n - \frac{1}{2}$ in which case only $l = j - \frac{1}{2} = n-1$ is possible.

For given j, $-j \leq m_j \leq j$ by integer steps, that is, $2j+1$ distinct values.

Therefore the highest sublevel $j = n - 1/2$ is degenerate $2j+1$ times. [1]

The other sublevels are degenerate $2(2j+1)$ times. [1]

C2

a) Proper time = time measured by a clock in the rest frame of a moving object [2]

Proper time between two events is the interval of time between the events in an reference frame in which they occur at the same space point (1)

$$d\tau = \frac{1}{c}\sqrt{dx^\mu dx_\mu} = \frac{1}{c}\sqrt{c^2dt^2 - d\underline{x}^2}$$

$$= dt\sqrt{1-\frac{1}{c^2}\left(\frac{d\underline{x}}{dt}\right)^2} = dt\sqrt{1-v^2/c^2}$$ [1]

$$\Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1-\underline{v}^2/c^2}} = \gamma$$ [2]

b) Four velocity $u^\mu = \frac{dx^\mu}{d\tau} =$

$$= \frac{dx^\mu}{dt}\frac{1}{\sqrt{1-\underline{v}^2/c^2}} = \left(\frac{c}{\sqrt{1-\underline{v}^2/c^2}}, \frac{\underline{v}}{\sqrt{1-\underline{v}^2/c^2}}\right)$$ [1]

$$\Rightarrow u^\mu u_\mu = \frac{c^2}{1-\underline{v}^2/c^2} - \frac{\underline{v}^2}{1-\underline{v}^2/c^2} = \frac{c^2-\underline{v}^2}{1-\underline{v}^2/c^2} = c^2$$ [2]

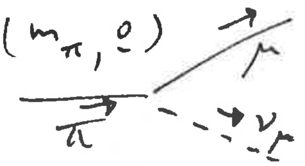
$p^\mu = m u^\mu = \left(\frac{E}{c}, \underline{p}\right)$ four-momentum

$p^\mu p_\mu = m^2 u^\mu u_\mu = m^2c^2 \Rightarrow$ (1)

$$\Rightarrow \quad \left(\frac{E}{c}\right)^2 - \underline{p}^2 = m^2 c^2 \qquad [1]$$

that is, $$E^2 = c^2 \underline{p}^2 + m^2 c^4 \qquad [1]$$

c) $P = (m_\pi, \underline{0})$ $\quad p = (E, \underline{p}), \; p^2 = m_\mu^2$ $\qquad (c = 1)$



p' $\qquad p'^2 \simeq 0$

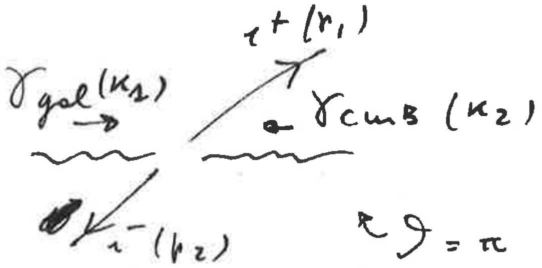
4-momentum conservation $\Rightarrow$ $(P - p)^2 = p'^2$ $\qquad [2]$

i.e.

$$m_\pi^2 + m_\mu^2 - 2 E m_\pi = 0$$

$$\Rightarrow \quad E = \frac{m_\pi^2 + m_\mu^2}{2 m_\pi} \quad \text{muon energy} \qquad [2]$$

d)



$$\underbrace{(k_1 + k_2)^2}_{2k_1k_2 \,=} = \underbrace{(p_1 + p_2)^2}_{2m^2 + 2p_1p_2 \geq 4m^2}$$

$$= 2 E_{gal} E_{CMB} (1 - \cos\vartheta) \leq 4 E_{gal} E_{CMB}$$

Thus $$E_{gal} \geq \frac{4m^2}{4E_{CMB}} = \frac{m^2}{E_{CMB}} \qquad [2]$$

So the threshold energy of the galactic photons is

$$E_{threshold} = \frac{m^2}{E_{CMB}} = \frac{0.511^2 (10^6)^2}{2.3 \; 10^{-4}} \, eV$$

$$= 1.14 \; 10^{15} \, eV \qquad [2]$$