
SEMESTER 2 EXAMINATIONS 2012-2013

BSc Synoptic Examination

DURATION 120 MINS

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*Answer **all** questions in **Section A** and two **and only two** questions in **Section B.***

***Section A** carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.*

***Section B** carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.*

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only University approved calculators may be used.

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Section A

A1. The Maxwell curl equation for magnetic field may be written

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

in SI units, where $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{j}$ is the free charge current density and $\boldsymbol{E}$ is the electric field intensity. Demonstrate the dimensional consistency of this equation by showing that each term has dimensions of [V s m^{-3}]. What physical quantity has dimensions of [V s]? [3]

A2. Consider a satellite of mass m, in circular orbit, radius a, around a planet of mass M. Find an expression for the total energy of the satellite as a function of a. A booster rocket fires briefly, imparting an impulse in the direction antiparallel to the instantaneous velocity of the satellite, and reducing its kinetic energy. Describe the subsequent motion of the satellite qualitatively. [4]

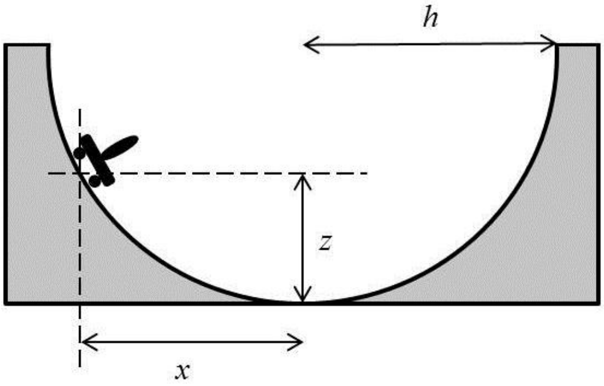
A3. An MRI (magnetic resonance imaging) body scanner uses a superconducting solenoid to generate an intense dc magnetic field within a cylindrical chamber large enough to accommodate a person. Estimate the magnitude in joules of the energy inside the chamber associated with the field when the body scanner is operating. [3]

A4. A reversible heat engine operates between a finite hot reservoir, initially at temperature T_1, and an infinite cold reservoir at constant temperature T_2. The heat capacity of the hot reservoir, C_0, is independent of temperature. Find an expression for the maximum amount of work that may be done by this engine. [5]

A5. Write an expression for the (un-normalised) wavefunction of a free particle of energy E, moving in 1 dimension along the positive x-axis in a region of zero potential. Now suppose that the particle encounters a negative potential step of height $-V_0$. Find an expression for the probability that the particle will be reflected by the step. [5]

SECTION B

B1.



The diagram represents a skateboarder of mass M rolling in a half-pipe with profile

$$z(x) = h - \sqrt{h^2 - x^2} \text{ for } |x| < h.$$

In this problem you may assume that the moment of inertia of the skateboarder about any axis through his or her centre of mass is negligible, and you may also neglect the distance between the centre of mass of the skateboarder and the surface of the half-pipe.

(i) The skateboarder drops into the half-pipe from point $(x, z) = (-h, h)$ with zero initial speed, and rolls to and fro without kicking. Find an expression for the speed of the board as a function of x, assuming that the wheel bearings are frictionless and that air resistance can be neglected. [4]

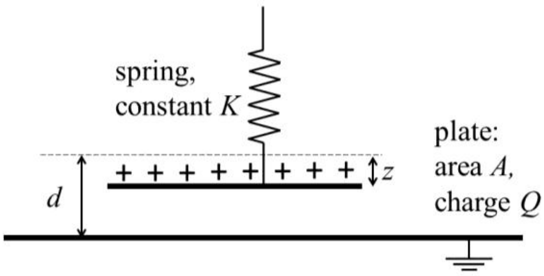
(ii) Find an expression for the magnitude of the force that the half-pipe instantaneously exerts on the skateboarder at $x = 0$. [4]

(iii) Explain why the angular momentum of the skateboarder is **not** conserved. [2]

(iv) The skateboarder kicks to slow down, and rolls to and fro in the bottom of the half-pipe. Show that, for sufficiently small amplitude, the skateboarder executes simple harmonic motion (SHM), with period T_0 that is independent of amplitude. Find an expression for T_0. [6]

(v) Determine from a diagram or otherwise whether the period of the skateboarder's motion increases or decreases as the amplitude of the motion increases outside the SHM regime, explaining your reasoning. (You do not need to find a general expression for the period.) [4]

B2



A capacitor, shown above, consists of a light flat conducting plate of area A, electrically isolated and suspended under vacuum by a spring above a conducting ground plane held at zero potential. The spring keeps the plate parallel to the ground plane and allows the separation between the plate and the ground plane to vary under the action of a linear restoring force of spring constant K.

(i) A charge $+Q$ is applied to the suspended plate. Explain why the charged plate is attracted towards the ground plane. [2]

(ii) State Gauss's theorem. Show, by reference to a suitable Gauss surface, that a uniform E-field of strength $E = Q/\varepsilon_0 A$ exists between the suspended plate and the ground plane, assuming that distortion of the field at the edges of the plate can be neglected. [2]

(iii) Write down an expression for the total potential energy U of the capacitor, with its suspended plate bearing charge Q and displaced by distance z from its equilibrium position. (The energy density of an electrostatic field in vacuum is $\varepsilon_0 E^2/2$.) [2]

(iv) Explain how the principle of virtual work leads to an expression for the total force F_z acting on the suspended plate. Use the result of (iii) to show that

$$F_z = -\frac{dU}{dz} = -Kz + \frac{Q^2}{2\varepsilon_0 A}.$$ [2]

(v) Show that in equilibrium the displacement of the capacitor plate is related to its charge by $z = Q^2/(2\varepsilon_0 AK)$. [1]

(vi) Prove that the potential difference between the suspended plate and the ground plane is given by

$$V = \frac{Q}{\varepsilon_0 A/d}\left(1 - \frac{Q^2}{2Kd\varepsilon_0 A}\right),$$

where d is the equilibrium distance between the uncharged plate and the ground plane. Sketch a graph of V against Q, explaining the range of Q over which this relationship is valid. [5]

(vii) Prove that the potential difference V between the suspended plate and the ground plane takes a maximum value when $Q = \sqrt{2\varepsilon_0 AKd/3}$ and $z = d/3$. [2]

(viii) Suppose that an external dc voltage source is connected across the capacitor. Explain why stable equilibrium states of the capacitor only exist for

$$-\sqrt{2\varepsilon_0 AKd/3} < Q < \sqrt{2\varepsilon_0 AKd/3}.$$

[4]

TURN OVER

B3. This question involves the Fermi-Dirac (F-D) distribution function describing the mean occupation number of a state of energy E for a fermion system in thermal equilibrium at temperature T with chemical potential $\mu(T)$:

$$\bar{n} = \frac{1}{1 + \exp\left(\frac{E-\mu}{k_B T}\right)}.$$

(i) State the Pauli Exclusion Principle. Describe one difference between fermions and bosons. [3]

(ii) Sketch a graph of $\bar{n}$ versus E corresponding to the absolute zero of temperature. Mark the Fermi energy, $E_F = \mu(0)$, on your graph, and explain its physical significance. [2]

(iii) Sketch a graph of $\bar{n}$ versus E corresponding to a temperature $T \ll E_F/k_B$. Explain with reference to your graph why only a fraction $\sim k_B T/E_F$ of fermions will contribute to thermodynamic properties such as heat capacity in this regime. [3]

(iv) Sketch a graph of $\bar{n}$ versus E corresponding to a temperature $T \gg E_F/k_B$. Explain why this graph describes a classical regime. [2]

(v) Consider a 3-dimensional Fermi gas of N free electrons occupying volume V, with $V/(2\pi)^3$ allowed k-vector values per unit volume of k-space. Show that the internal energy of this gas in its ground state at $T = 0$ is given by

$$U = \frac{3}{5} N \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V} \right)^{\frac{2}{3}}.$$

[5]

(vi) Use classical thermodynamic arguments to justify the relation

$$dU = -p dV$$

for an infinitesimal change in a fixed mass of gas at temperature $T = 0$ and pressure p. Hence derive an expression for the pressure exerted by the Fermi gas described in (v) at $T = 0$. [3]

(vii) Contrast the behaviour of the Fermi gas described in (v) with that of a classical ideal gas. [2]

B4. The tip of a nanostructured silicon cantilever is free to oscillate in 1 dimension, with displacement x and momentum p, under the action of a harmonic restoring force with spring constant K_0. The Hamiltonian operator for this system may be written

$$H = \frac{p^2}{2M} + \frac{1}{2} K_0 x^2$$

where M is the effective mass. Note the value of the definite integral:

$$\int_{-\infty}^{+\infty} \exp(-Bx^2) dx = \sqrt{\frac{\pi}{B}}.$$

(i) Write down the time-independent Schrödinger equation for the silicon cantilever system. [1]

(ii) Show that the time-independent Schrödinger equation in (i) has a solution of the form

$$u_0(x) = A_0 \exp\left(-\frac{1}{2}\alpha_0^2 x^2\right)$$

where A_0 is a real constant and $\alpha_0^4 = MK_0/\hbar^2$. Find the energy eigenvalue of the cantilever in this state. [3]

(iii) Explain why the wavefunction of the cantilever must be normalised to represent its state correctly. State the normalisation condition, and determine the value of A_0 that normalises $u_0(x)$ defined in (ii). [3]

(iv) Explain why it is necessary to cool the cantilever to a temperature $T < T_0$ in order to observe non-classical effects. Estimate a value for T_0, given that

$$\frac{1}{2\pi}\sqrt{\frac{K_0}{M}} = 3\ \mathrm{GHz}.$$

[2]

(v) There occurs a sudden change in the internal structure of the cantilever, after which its spring constant takes the weaker value

$$K = \left(\frac{5}{12}\right)^4 K_0,$$

with no change in effective mass or equilibrium position. Immediately after the change, the wavefunction of the cantilever is $u_0(x)$, given in (ii). Explain why the cantilever is **not** now in a *stationary state*. [4]

(vi) Show that after the change described in (v) a measurement of the energy of the cantilever will yield the value

$$\frac{1}{2}\hbar\sqrt{\frac{K}{M}}$$

with probability $\frac{120}{169}$. [5]

(vii) Suggest other possible outcomes of the energy measurement described in (vi). [2]

END OF PAPER