SEMESTER 1 EXAMINATION 2013-2014

ADVANCED QUANTUM PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 8 questions.

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. Consider the following four expressions, where $\hat{A}$ is an operator and $|\psi\rangle$, $|\phi\rangle$ are arbitrary kets:

$$\text{(i)}\quad \langle\psi|\hat{A}|\phi\rangle\langle\psi|\phi\rangle\ , \quad \text{(ii)}\quad \langle\psi|\phi\rangle\langle\psi|\hat{A}\ , \quad \text{(iii)}\quad \langle\psi|\phi\rangle\hat{A}|\phi\rangle\langle\psi|\ , \quad \text{(iv)}\quad \hat{A}|\psi\rangle\langle\phi|\hat{A}|\psi\rangle\ .$$

(a) For each expression, state whether it is a scalar, operator, ket or bra. [2]

(b) Obtain the adjoint of each expression. [4]

A2. Show that any operator $\hat{\Omega}$ can be written as $\sum_{i,j=1}^{n} \Omega_{ij}|i\rangle\langle j|$, where Ω_{ij} are the matrix elements of $\hat{\Omega}$, and $\{|i\rangle\}$ and $\{|j\rangle\}$ are the same orthonormal basis with basis vectors labelled by the indices i and j respectively. [4]

A3. If two Hermitian operators $\hat{A}$ and $\hat{B}$ have the same eigenvalues with the same multiplicities, show that $\hat{A} = U^{\dagger}\hat{B}U$ for some unitary operator U. [3]

A4. Let $|x\rangle$ and $|p\rangle$ be eigenstates of position and momentum respectively, which satisfy $\hat{x}|x\rangle = x|x\rangle$ and $\hat{p}|p\rangle = p|p\rangle$, and $\langle p|x\rangle = \exp(-ipx/\hbar)/\sqrt{2\pi\hbar}$. By considering $\langle p|\hat{x}|\psi\rangle$ for an arbitary state $|\psi\rangle$, find the explicit form of the operator $\hat{x}$ in momentum space. [3]

A5. An orthonormal basis of a Hamiltonian operator is defined as follows:

$$H|1\rangle = E|1\rangle, \quad H|2\rangle = 2E|2\rangle, \quad H|3\rangle = 3E|3\rangle, \quad H|4\rangle = 4E|4\rangle\ .$$

A system is in the state

$$|\psi\rangle = 3|1\rangle + |2\rangle - |3\rangle + 7|4\rangle\ .$$

(a) If a measurement of the energy of this state is made, what results can be found and with what probabilities? [3]

(b) Find the average energy of the system. Comment on whether or not this value can ever be obtained from an energy measurement of this system. [1]

Section B

B1. The operator $\hat{\mathbf{S}}$, which represents the spin of a spin-$\frac{1}{2}$ particle (such as the electron) may be written as $\hat{\mathbf{S}}$=$(\hat{S}_x, \hat{S}_y, \hat{S}_z)$ where

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

are matrix representations of the (x, y, z) components of the spin operator.

(a) Obtain the matrix representation of the operator for the total spin of the particle, $\hat{S}^2$. [2]

(b) What are the eigenvectors and eigenvalues of $\hat{S}^2$? [2]

(c) Explain whether or not the angular momentum vector can be aligned with a given axis (i.e. the x, y, or z axis). [2]

(d) Briefly describe two differences between intrinsic angular momentum $\hat{\mathbf{S}}$ and orbital angular momentum $\hat{\mathbf{L}}$. [4]

(e) By explicitly calculating the expectation values of $\hat{S}_x$, $\hat{S}_y$ and $\hat{S}_z$ (given by $\langle S_x \rangle$, $\langle S_y \rangle$ and $\langle S_z \rangle$ respectively), show that it is impossible for a particle to be in a state

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

such that

$$\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0\,.$$ [10]

B2. A system contains two distinguishable particles with individual angular momentum operators $\hat{J}^{(1)}$ and $\hat{J}^{(2)}$, respectively. Setting $\hbar = 1$, the components of these two operators obey the following commutation relations:

$$[\hat{J}_i^{(1)}, \hat{J}_j^{(1)}] = i\epsilon_{ijk}\hat{J}_k^{(1)} \ , \quad [\hat{J}_i^{(2)}, \hat{J}_j^{(2)}] = i\epsilon_{ijk}\hat{J}_k^{(2)} \ , \quad [\hat{J}_i^{(1)}, \hat{J}_j^{(2)}] = 0 \ ,$$

where ϵ_{ijk} changes sign under interchange of any two indices and $\epsilon_{123} = 1$.

(a) Define the operator for total angular momentum, $\hat{J}$, for the two-particle system and write down the commutation relations between its components. [2]

(b) Prove that the four operators $\hat{J}^2$, $\hat{J}_3$, $(\hat{J}^{(1)})^2$, and $(\hat{J}^{(2)})^2$ form a commuting set of operators. [8]

(c) If both particles are spin-$\frac{1}{2}$ particles, by making use of the relation

$$\hat{J}_-|j, m\rangle = \sqrt{(j+m)(j-m+1)}|j, m-1\rangle \ ,$$

write down all the simultaneous eigenstates of $\hat{J}^2$ and $\hat{J}_3$ as linear superpositions of the eigenstates of $(\hat{J}^{(1)})^2$, $\hat{J}_3^{(1)}$, $(\hat{J}^{(2)})^2$ and $\hat{J}_3^{(2)}$. [10]

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B3. (a) Write down an expression for the most general state ($|\psi\rangle_{AB}$) in a Hilbert space $V_A \otimes V_B$, where V_A and V_B are Hilbert spaces with orthonormal bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ respectively. Then give the condition for this general state to be separable and the condition for this state to be entangled. [4]

(b) In an EPR type experiment with a spinless particle (i.e. zero intrinsic angular momentum) decaying to electrons A and B, the state of the two electrons is given by the superposition:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B) \ ,$$

where $|\uparrow\rangle_A$ ($|\downarrow\rangle_A$) represents particle A in an eigenstate with $\hat{S}_z$ eigenvalue $\hat{h}/2$ ($-\hat{h}/2$), and likewise for particle B. The experiment is arranged such that the spin of particle A is measured in the positive z direction, while the spin of particle B is measured in a direction in the xz plane at an angle θ_{ab} to the z axis. The parameters a_n and b_n are the results of the individual measurements of the spin in the given directions for particles A and B respectively, with a scaling such that $a_n = \pm 1$ and $b_n = \pm 1$. The expectation value for ab is given by:

$$\langle ab \rangle = \left(\frac{2}{\hbar}\right)^2 \langle \psi | \hat{S}_{Az} \otimes (\hat{S}_{Bz} \cos\theta_{ab} + \hat{S}_{Bx} \sin\theta_{ab}) | \psi \rangle \ .$$

Here $\hat{S}_{Az}$ ($\hat{S}_{Bz}$) is the spin operator for particle A (B) in the z direction, and $\hat{S}_{Bx}$ is the spin operator for particle B in the x direction. Show that $\langle ab \rangle = -\cos\theta_{ab}$. [12]

(c) For measurements of the spin in two other directions, with results a'_n and b'_n for particle A and B respectively, and an experimental setup such that θ_{ab}=0, $\theta_{ab'} = \theta_{a'b} = \phi$, and $\theta_{a'b'} = 2\phi$, write down the expression for the quantity $|\langle g \rangle| = |\langle ab \rangle + \langle a'b \rangle + \langle ab' \rangle - \langle a'b' \rangle|$. State (without deriving) an upper bound for $|\langle g \rangle|$ in a theory which assumes both local realism and objective reality, and compare with the quantum mechanical prediction for $|\langle g \rangle|$. Comment on which theory agrees with the results of experiments which have carried out the measurement of $|\langle g\rangle|$. [4]

END OF PAPER