Section A Solutions to Semester 1 exam
PHYS 6003, 2014/2015

(Bookwork)

A1) $a_1^* |v\rangle + a_2 |w\rangle\langle p|q\rangle + a_3 \hat{\Omega}\hat{\Lambda}|r\rangle + a_4 \hat{\Lambda}|u\rangle$

$\downarrow$

$\langle v| a_1 + \langle q|p\rangle\langle w| a_2^* + \langle r|\hat{\Lambda}^\dagger\hat{\Omega}^\dagger a_3^* + \langle u|\hat{\Lambda}^\dagger a_4^*$

[1/2] mark [1/2] mark [1/2] mark [1/2] mark

Total [2] marks

(Bookwork)

A2) Multiply both sides from left by $\langle j|$ where j takes a particular value from 1 to n [1/2] mark

$$\Rightarrow \langle j|u\rangle = \sum_{i=1}^{n} u_i \langle j|i\rangle = \sum_{i=1}^{n} u_i \delta_{ji} = u_j$$

[1/2] mark [1/2] mark [1/2] mark [1/2] mark

and swap j for i so $u_i = \langle i|u\rangle$ [1/2] mark

$$\text{Now } |u\rangle = \sum_{i=1}^{n} \langle i|u\rangle |i\rangle = \sum_{i=1}^{n} |i\rangle\langle i|u\rangle = \left(\sum_{i=1}^{n} |i\rangle\langle i|\right)|u\rangle$$

[1/2] mark [1/2] mark [1] mark

This is true for any ket $|u\rangle$

and so $I = \sum_{i=1}^{n} |i\rangle\langle i|$

[1] mark

Total [6] marks

A3) (Bookwork)

Consider the following equation for the orthogonal set of vectors $|i\rangle$

$$\sum_{i=1}^{n} a_i |i\rangle = |0\rangle \qquad [1] \text{ mark}$$

Multiply from Left-hand side by $\langle j|$ for a particular value of j

$$\Rightarrow \sum_{i=1}^{n} a_i \langle j|i\rangle = 0 \qquad [1] \text{ mark}$$

since $\langle j|i\rangle = 0$ for $i \neq j$ [½] mark

and $\langle j|i\rangle \neq 0$ for $i = j$ [½] mark

then $a_j = 0$ for $i = j$ [1] mark

This can be done for all choices of j from 1 to n, [½] mark and so $a_j = 0$ for all j.

[½] mark

$\Rightarrow$ only solution to $\sum_{i=1}^{n} a_i |i\rangle = |0\rangle$

is $a_i = 0$ for all i.

[½] mark

and so $\{|i\rangle\}$ are linearly independent vectors

[½] mark

Total [6] marks

(Bookwork)

A4) Require that the wavefunction be single-valued for a given value of ϕ [1] mark

$$\Rightarrow \quad e^{im\phi} = e^{im(\phi+2\pi)}$$

and so $e^{im2\pi} = 1$ [1] mark

$\Rightarrow$ m can only take integer values

[1] mark

Total [3] marks

(Bookwork)

A5) A classical bit takes a value of either 0 or 1 [½] mark. A qubit is either $|0\rangle$, $|1\rangle$, or a superposition $\alpha|0\rangle + \beta|1\rangle$,

[½] mark [½] mark [½] mark

where $|0\rangle$ and $|1\rangle$ form an orthonormal basis.

[½] mark

An example of a qubit is a state vector of quantum mechanical spin [½] mark of a spin ½ particle

(or polarisation states of a photon)

Total [3] marks

Total [2]+[6]+[6]+[3]+[3]

= [20] marks

for section A

Section B

B1)

a) $[\hat{H}, a] = [\hbar\omega(a^\dagger a + \tfrac{1}{2}), a] = \hbar\omega(\underbrace{[\tfrac{1}{2}, a]}_{=0} + [a^\dagger a, a])$ (1)

Based on problem sheets + past exam

$= \hbar\omega(a^\dagger a a - a a^\dagger a) = \hbar\omega[a^\dagger, a]a = -\hbar\omega a$ (1)

$[\hat{H}, a^\dagger] = [\hbar\omega(a^\dagger a + \tfrac{1}{2}), a^\dagger] = \hbar\omega(\underbrace{[\tfrac{1}{2}, a^\dagger]}_{=0} + a^\dagger a, a^\dagger)$ (1)

$= \hbar\omega(a^\dagger a a^\dagger - a^\dagger a^\dagger a) = \hbar\omega a^\dagger[a, a^\dagger] = \hbar\omega a^\dagger$ (1)

Total [4] marks for a)

b) $\hat{H}(a^\dagger|n\rangle) = (\hat{H}a^\dagger)|n\rangle = (a^\dagger\hat{H} + \hbar\omega a^\dagger)|n\rangle$

$= a^\dagger\hat{H}|n\rangle + \hbar\omega a^\dagger|n\rangle$ (1)

Based on problem sheets + past exam

now $a^\dagger\hat{H}|n\rangle = a^\dagger(E_n|n\rangle) = E_n a^\dagger|n\rangle$

so $\hat{H}(a^\dagger|n\rangle) = E_n a^\dagger|n\rangle + \hbar\omega a^\dagger|n\rangle = (E_n + \hbar\omega)a^\dagger|n\rangle$

so $a^\dagger|n\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $(E_n + \hbar\omega)$ (1)

$\hat{H}(a|n\rangle) = (\hat{H}a)|n\rangle = (a\hat{H} - \hbar\omega a)|n\rangle = a\hat{H}|n\rangle - \hbar\omega a|n\rangle$ (1)

Now $a\hat{H}|n\rangle = a(E_n|n\rangle) = E_n(a|n\rangle)$

and so $\hat{H}(a|n\rangle) = E_n a|n\rangle - \hbar\omega a|n\rangle = (E_n - \hbar\omega)(a|n\rangle)$

$\Rightarrow a|n\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $(E_n - \hbar\omega)$ [1]

Total [4] marks for b)

c) write $\hat{p} = -i\sqrt{\frac{m\hbar\omega}{2}}\,(a - a^\dagger)$ [1/2] mark

(Based on problem sheets)

$$\hat{p}^2 = \frac{m\hbar\omega}{2}\left(-a^2 + aa^\dagger + a^\dagger a - a^\dagger a^\dagger\right) \text{ [1/2] mark}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger) \text{ [1/2] mark}$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(a^2 + aa^\dagger + a^\dagger a + a^\dagger a^\dagger\right) \text{ [1/2] mark}$$

subtotal [2] marks

$$\text{Now } \langle p\rangle = \langle n|\hat{p}|n\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\langle n|a - a^\dagger|n\rangle \text{ [1/2] mark}$$

$$= -i\sqrt{\frac{m\hbar\omega}{2}}\Big(\sqrt{n}\underbrace{\langle n|n-1\rangle}_{=0} - \sqrt{n+1}\underbrace{\langle n|n+1\rangle}_{=0}\Big) \text{ [1/2] mark}$$

by orthonormality

$$\text{Now } \langle p^2\rangle = \langle n|\hat{p}^2|n\rangle = \frac{m\hbar\omega}{2}\langle n|-a^2 + aa^\dagger + a^\dagger a - a^\dagger a^\dagger|n\rangle$$

$$= \frac{m\hbar\omega}{2}\langle n|-a^2 + 2a^\dagger a + 1 - a^\dagger a^\dagger|n\rangle \text{ [1/2]}$$

$$= \frac{m\hbar\omega}{2}\Big(\underbrace{-\langle n|aa|n\rangle}_{(i)} + 2\langle n|a^\dagger a|n\rangle + \langle n|n\rangle - \underbrace{\langle n|a^\dagger a^\dagger|n\rangle}_{(iv)}\Big)$$

Terms (i) and (iv) vanish by orthonormality [1/2] mark

$$\Rightarrow \langle p^2\rangle = \frac{m\hbar\omega}{2}\left(1+2n\langle n|n\rangle\right)$$ [1/2] mark

and so $\langle p^2\rangle = m\hbar\omega\left(\tfrac{1}{2}+n\right)$ [1/2] mark

subtotal [3]

$$\langle x\rangle = \langle n|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a+a^\dagger|n\rangle$$ [1/2] mark

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle n|n-1\rangle+\sqrt{n+1}\langle n|n+1\rangle\right)$$

$= 0$ by orthonormality [1/2] mark

$$\langle x^2\rangle = \langle n|\hat{x}^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|a^2+aa^\dagger+a^\dagger a+a^\dagger a^\dagger|n\rangle$$ [1/2] mark

gives same result as $\langle p^2\rangle$ since only differences are the signs of a^2 and $a^\dagger a^\dagger$, and these terms vanish by orthonormality [1/2] mark

subtotal [2]

$$\Delta p = \sqrt{\langle p^2\rangle-\langle p\rangle^2} = \sqrt{\frac{m\omega\hbar(2n+1)}{2}}$$

$$\Delta x = \sqrt{\langle x^2\rangle-\langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}(2n+1)}$$

$$\Rightarrow \Delta x\Delta p = \frac{\hbar}{2}(2n+1)$$ [1/2] mark

Heisenberg's uncertainty relation is always satisfied because $n\geq 0$. For $n=0$ one has $\Delta x\Delta p = \frac{\hbar}{2}$ [1/2] mark

subtotal [1]

Total [8] marks for c)

d) $\langle n|\hat{x}^2|m\rangle = \frac{\hbar}{2m\omega}\langle n|a^2 + aa^\dagger + a^\dagger a + a^\dagger a^\dagger|m\rangle$

(unseen)

Now is $n = m-2$ then a^2 term survives since $\langle n|aa|n+2\rangle \sim \langle n|n\rangle$ while other terms vanish by orthonormality $\Rightarrow \langle n|\hat{x}^2|m\rangle \neq 0$ for $n = m-2$

[1] mark

If $n = m+2$ then $a^\dagger a^\dagger$ term survives while others vanish: $\Rightarrow \langle n|\hat{x}^2|m\rangle \neq 0$ for $n = m+2$

[1] mark

If $n = m$ we have $\langle n|x^2|m\rangle \neq 0$ because $aa^\dagger$ and $a^\dagger a$ terms survive (see part c)

[1] mark

For all other choices of n and m one has $\langle n|\hat{x}^2|m\rangle = 0$ because no terms of the form $\langle n|n\rangle$ survive

[1] mark

Total [4] marks

Total for B1 = [4]+[4]+[8]+[4] = [20] marks

B2) (Bookwork)

To find the matrix representation of $\hat{L}_x$ we need to compute the effect of $\hat{L}_x$ on the basis vectors $|1\rangle_z, |0\rangle_z, |-1\rangle_z$ (1) mark

$$\hat{L}_x|1\rangle_z = \tfrac{1}{2}(\hat{L}_+ + \hat{L}_-)|1\rangle_z = \tfrac{1}{2}L_-|1\rangle_z = \frac{\hbar}{\sqrt{2}}|0\rangle_z$$ (1) mark

$$\hat{L}_x|0\rangle_z = \tfrac{1}{2}(\hat{L}_+ + \hat{L}_-)|0\rangle_z = \frac{\hbar}{\sqrt{2}}(|1\rangle_z + |-1\rangle_z)$$ (1) mark

$$\hat{L}_x|-1\rangle_z = \tfrac{1}{2}(\hat{L}_+ + \hat{L}_-)|-1\rangle_z = \tfrac{1}{2}L_+|-1\rangle_z = \frac{\hbar}{\sqrt{2}}|0\rangle_z$$ (1) mark

Now matrix representation is given by:

$$(\hat{L}_x)_{ij} = \langle i|\hat{L}_x|j\rangle$$ (1) mark

with $i,j = 1,0,-1$

(1) mark

using orthonormality e.g. $\langle 1|1\rangle = 1$, $\langle 0|1\rangle = 0$ etc

one has

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$ (1) mark

Total [7] marks for a)

b) Obtain characteristic polynomial from $\det(\hat{L}_x - \lambda I) = 0$ where λ are eigenvalues. (1) mark

(Based on problem sheets + past exam)

Let $c = \frac{\hbar}{\sqrt{2}}$

$$\det\begin{bmatrix} -\lambda & c & 0 \\ c & -\lambda & c \\ 0 & c & -\lambda \end{bmatrix} = -\lambda(\lambda^2 - c^2) - c(-c\lambda) + 0 = 0$$

1 mark

$$\Rightarrow -\lambda^3 + \lambda c^2 + \lambda c^2 = 0$$
$$\Rightarrow \lambda(2c^2 - \lambda^2) = 0$$

so $\lambda = 0$ or $\lambda = \pm\sqrt{2}c = \pm\hbar$ since $c = \frac{\hbar}{\sqrt{2}}$

1 mark

Let column vector representing $|1\rangle_x$ be $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ 1/2 mark

where $a^2 + b^2 + c^2 = 1$ is normalisation condition

1/2 mark

Therefore $\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \hbar\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ 1/2 mark

so $b = \sqrt{2}a$
$a + c = \sqrt{2}b$ 1/2 mark and so $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \begin{pmatrix} a \\ \sqrt{2}a \\ a \end{pmatrix}$ 1/2 mark
$b = 2\sqrt{c}$

normalisation condition gives $a^2 + 2a^2 + a^2 = 1$
so $a = 1/2$ (ignore phase) and $|1\rangle_x = 1/2\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$

1/2 mark

For eigenvector $|0\rangle_x$ one has

$$\frac{\hbar}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$ [½] marks

so $b=0$ and $a+c=0$ [½] mark and hence

$$\begin{pmatrix}a\\ b\\ c\end{pmatrix} \rightarrow \begin{pmatrix}a\\ 0\\ -a\end{pmatrix}$$ [½] mark

normalisation gives $a^2+0^2+a^2=1$

and so $a = \frac{1}{\sqrt{2}}$ (ignore phase)

and $|0\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$ [½] mark

For eigenvector $|-1\rangle_x$ one has

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix} = -\hbar\begin{pmatrix}a\\ b\\ c\end{pmatrix}$$ [½] mark

so $b=-\sqrt{2}a$, $a+c=-\sqrt{2}b$, $b=-\sqrt{2}c$ [½] mark

and so $\begin{pmatrix}a\\ b\\ c\end{pmatrix} \rightarrow \begin{pmatrix}a\\ -\sqrt{2}a\\ a\end{pmatrix}$ [½] mark.

Normalisation gives $a^2+2a^2+a^2=1$ and so $a=\frac{1}{2}$ (ignore phase)

and $|-1\rangle_x = \frac{1}{2}\begin{pmatrix}1\\ -\sqrt{2}\\ 1\end{pmatrix}$ [½] mark

Total [10] marks for b)

c) One needs to calculate $|\langle 0_x|\psi\rangle|^2$ [1] mark

(Based on marker's sheets)

$$= \left| \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \frac{1}{\sqrt{14}} \right|^2$$ [1] mark

$$= \left| \left(\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}}\right) \frac{1}{\sqrt{14}} \right|^2 = \left| \frac{2}{\sqrt{28}} \right|^2 = \frac{4}{28} = \frac{1}{7}$$

$\Rightarrow$ probability of measuring $\hat{L}_x = 0$ is $\frac{1}{7}$

[1] mark

Total [3] marks for c)

Total for B2: [7] + [10] + [3] = [20] marks

B3) (All bookwork)

i) The state of a particle is represented by a vector $|\psi(t)\rangle$, the state vector, in Hilbert space. [1] mark

ii) The independent variables x and p of classical mechanics now become Hermitian operators, $\hat{x}$ and $\hat{p}$ respectively, defined by the commutator $[\hat{x}, \hat{p}] = i\hbar$ [1] mark.

Dependent variables $\omega(p, x)$ in classical mechanics are given by the Hermitian operators

$$\Omega(\hat{x}, \hat{p}) = \omega(x \to \hat{x}, p \to \hat{p})$$ [1] mark

iii) If a particle is in a state $|\psi\rangle$, a measurement of the observable corresponding to Ω will yield one of the eigenvalues ω_i with probability $\text{Prob}(\omega_i) \propto |\langle \omega_i | \psi \rangle|^2$, where $|\omega_i\rangle$ is the eigenvector with eigenvalue ω_i. [1] mark. The state of the system will change from $|\psi\rangle$ to $|\omega_i\rangle$ as a result of the measurement [1] mark

iv) $|\psi(t)\rangle$ satisfies the Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

P.T.O

where $\hat{H}$ is the Hamiltonian (1) mark

Total [6] marks

b)

$$\langle\hat{\Omega}\rangle = \sum_i P(\omega_i)\omega_i = \sum_i |\langle\omega_i|\psi\rangle|^2\omega_i = \sum_i \langle\psi|\omega_i\rangle\langle\omega_i|\psi\rangle\omega_i$$

(1) mark (1) mark

Now since $\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle$ for each i

then $\langle\hat{\Omega}\rangle = \sum_i \langle\psi|\Omega|\omega_i\rangle\langle\omega_i|\psi\rangle$

(1) mark

Use $\sum_i |\omega_i\rangle\langle\omega_i| = I$ (1) mark to get $\langle\hat{\Omega}\rangle = \langle\psi|\Omega|\psi\rangle$

(1) mark

Total [5] marks

c)

The eigenvectors of a Hermitian operator in a Hilbert space form a basis for the space called an "eigenbasis" (1) mark

since $\Omega(\alpha|\omega_1\rangle + \beta|\omega_2\rangle) = \omega(\alpha|\omega_1\rangle + \beta|\omega_2\rangle)$ for arbitrary scalars α and β (with $\alpha^2+\beta^2=1$ for normalisation) (1) mark, then any linear combination $\alpha|\omega_1\rangle + \beta|\omega_2\rangle$ ~~is~~ ~~are~~ and its orthogonal combination are eigenvectors with eigenvalue ω (1) mark, and so there exists an infinity of eigenbases [1] mark. Total [4] marks

d) One has $\Lambda(\Omega|\lambda_i\rangle) = \Omega\Lambda|\lambda_i\rangle = \Omega\lambda_i|\lambda_i\rangle$ equation 1

[1/2] mark [1/2] mark eqn (1)

$= \lambda_i(\Omega|\lambda_i\rangle)$

[1/2] mark

$\Rightarrow (\Omega|\lambda_i\rangle)$ is an eigenvector of Λ

[1/2] mark

Now we know that $\Omega|\lambda_i\rangle \neq |u\rangle$ i.e. Ω cannot change $|\lambda_i\rangle$ to a different ket $|u\rangle$

[1] mark

If this were true then equation 1 above would give $\Lambda|u\rangle = \lambda_i|u\rangle$, which contradicts our assumption that $|\lambda_i\rangle$ is the only ket with eigenvalue λ_i [1] mark

$\Rightarrow \Omega|\lambda_i\rangle = \omega_i|\lambda_i\rangle$ is the only possibility, and so $\{|\lambda_i\rangle\}$ are also eigenvectors of Ω [1] mark

Total [5] marks

Total for B3: [6]+[5]+[4]+[5]=[20] marks

B4)

(Bookwork, seen in past exam)

a) Most general state is $|\psi\rangle_{AB} = \sum_{i,j} a_{ij} |i\rangle_A \otimes |j\rangle_B$

where a_{ij} are scalars [1]

The state is separable if $a_{ij} = a_i^A a_j^B$ [1]

where $|\psi\rangle_A = \sum_i a_i^A |i\rangle_A$ and $|\psi\rangle_B = \sum_j a_j^B |j\rangle_B$ [1]

The state is entangled if $a_{ij} \neq a_i^A a_j^B$ [1]

Total [4] marks for a)

b, c and d
(Bookwork, seen in past exam)

b) Now $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi\rangle = |B_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$\Rightarrow |\phi\rangle|\psi\rangle = \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \beta|100\rangle + \alpha|011\rangle + \beta|111\rangle\right)$$

[½] mark for each term = [2] marks

Now use $|00\rangle = \frac{1}{\sqrt{2}}\left(|B_0\rangle + |B_2\rangle\right)$

$|01\rangle = \frac{1}{\sqrt{2}}\left(|B_1\rangle + |B_3\rangle\right)$ [½] mark each = [2] marks

$|11\rangle = \frac{1}{\sqrt{2}}\left(|B_0\rangle - |B_2\rangle\right)$

$|10\rangle = \frac{1}{\sqrt{2}}\left(|B_1\rangle - |B_3\rangle\right)$

so $$|\phi\rangle|\psi\rangle = \frac{1}{2}\Big(\alpha|B_0\rangle|0\rangle + \alpha|B_2\rangle|0\rangle + \beta|B_1\rangle|0\rangle - \beta|B_3\rangle|0\rangle + \alpha|B_1\rangle|1\rangle + \alpha|B_3\rangle|1\rangle + \beta|B_0\rangle|1\rangle - \beta|B_2\rangle|1\rangle\Big)$$

[1] mark

and so

$$2|\phi\rangle|\psi\rangle = \Big(|B_0\rangle(\alpha|0\rangle+\beta|1\rangle) + |B_1\rangle(\alpha|1\rangle+\beta|0\rangle) + |B_2\rangle(\alpha|0\rangle-\beta|1\rangle) + |B_3\rangle(\alpha|1\rangle-\beta|0\rangle)\Big)$$

[1] mark

Total [6] marks for b)

c) $$\hat{I}(\alpha|0\rangle+\beta|1\rangle) = (|0\rangle\langle 0| + |1\rangle\langle 1|)(\alpha|0\rangle+\beta|1\rangle) = \alpha|0\rangle+\beta|1\rangle = |\phi\rangle$$

by orthonormality $\langle 0|0\rangle = \langle 1|1\rangle = 1$, $\langle 0|1\rangle = \langle 1|0\rangle = 0$

$$\hat{X}(\alpha|0\rangle+\beta|1\rangle) = (|0\rangle\langle 1| + |1\rangle\langle 0|)(\alpha|1\rangle+\beta|0\rangle) = \alpha|0\rangle+\beta|1\rangle = |\phi\rangle$$

$$\hat{Y}(\alpha|1\rangle-\beta|0\rangle) = (|0\rangle\langle 1| - |1\rangle\langle 0|)(\alpha|1\rangle-\beta|0\rangle) = \alpha|0\rangle+\beta|1\rangle = |\phi\rangle$$

$$\hat{Z}(\alpha|0\rangle-\beta|1\rangle) = (|0\rangle\langle 0| - |1\rangle\langle 1|)(\alpha|0\rangle-\beta|1\rangle) = \alpha|0\rangle+\beta|1\rangle = |\phi\rangle$$

[1] mark each

Total [4] marks for c)

d) To use these results for teleportation:

1) Alice has a qubit in an unknown state $|\phi\rangle$ i.e. α and β are unknown in $\alpha|0\rangle + \beta|1\rangle$ ½ mark

2) Alice keeps one of the qubits in the entangled state $|\psi\rangle$ and sends the other qubit to Bob. ½ mark

3) In the product state $|\phi\rangle|\psi\rangle$, the first two qubits (which are with Alice) are expanded in terms of Bell states, giving the expression for $|\phi\rangle|\psi\rangle$ in b) ½ mark

4) Alice measures which Bell state her two qubits are in, ½ mark collapsing $|\phi\rangle|\psi\rangle$ to the form $|B_i\rangle \otimes$ single qubit part. ½ mark Bob's qubit will now be in the state given by the single qubit part ½ mark

5) If Alice measures $|B_i\rangle$ ½ mark she sends the number i ($i = 1, 2, 3$ or 0) to Bob using two bits of classical information. ½ mark

6) Bob applies one of the operators to his qubit, according to whether Alice sends him 0, 1, 2 or 3. ½ mark

Bob applies	$\hat{I}$	if	Alice	sends	number	0
" "	$\hat{X}$	"	"	"	"	1
" "	$\hat{Z}$	"	"	"	"	2
" "	$\hat{Y}$	"	"	"	"	3

½ mark

The results in c) show that this operation transforms Bob's qubit into $|\phi\rangle$. ½ mark

7) $|\phi\rangle$ is transported from Alice to Bob without either of them knowing the state $|\phi\rangle$.

[1/2] mark

Total [6] marks for part d)

Total for B4

[4] + [6] + [4] + [6] = [20] marks