SEMESTER 1 EXAMINATION 2014-2015

ADVANCED QUANTUM PHYSICS

Duration: 120 MINS (2 hours)

This paper contains 9 questions.

Answer **all** questions in **Section A** and **only two** questions in **Section B**.

**Section A** carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

**Section B** carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

## Section A

**A1.** Write down the adjoint of the following expression involving bras and kets, in which $a_i$ are complex scalars, and $\hat{\Omega}$ and $\hat{\Lambda}$ are operators:

$$a_1^*|v\rangle + a_2|w\rangle\langle p|q\rangle + a_3\hat{\Omega}\hat{\Lambda}|r\rangle + a_4\hat{\Lambda}|u\rangle\,.$$ [ 2 ]

**A2.** Consider the arbitrary ket $|u\rangle = \sum_{i=1}^{n} u_i|i\rangle$, where $\{|i\rangle\}$ is an orthonormal basis. Show that $u_i = \langle i|u\rangle$ for all values of $i$. Using this result, then prove that $\sum_{i=1}^{n} |i\rangle\langle i| = \hat{I}$, where $\hat{I}$ is the identity operator. [ 6 ]

**A3.** The kets in the set $\{|i\rangle\}$ have non-zero (and finite) norm, and are mutually orthogonal (i.e. $\langle i|j\rangle = 0$ for $i \neq j$). Show that they form a set of linearly independent vectors. [ 6 ]

**A4.** The $\phi$ dependent part of the position space wavefunction (in spherical co-ordinates) of an eigenvector of orbital angular momentum is given by $e^{im\phi}$, where $m$ is the quantum number associated with the $z$ component of orbital angular momentum. Show that $m$ has to be an integer. [ 3 ]

**A5.** Explain the difference between a classical bit and a qubit, giving an example of the latter. [ 3 ]

## Section B

**B1.** A simple harmonic oscillator in one dimension is defined by the Hamiltonian

$$\hat{H} = \hbar\omega(a^\dagger a + 1/2)\,,$$

where the raising and lowering operators are given respectively by

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}\,,$$
$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}\,,$$

with $[\hat{x}, \hat{p}] = i\hbar$ and $[a, a^\dagger] = 1$. Let $|n\rangle$ be a normalised (i.e. $\langle n|n\rangle = 1$) eigenvector of the Hamiltonian with an energy eigenvalue of $E_n$ (i.e. $\hat{H}|n\rangle = E_n|n\rangle$).

(a) Show that $[\hat{H}, a] = -\hbar\omega a$ and $[\hat{H}, a^\dagger] = \hbar\omega a^\dagger$. [ 4 ]

(b) Using the commutation relations in (a), show that $a|n\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $E_n - \hbar\omega$, and that $a^\dagger|n\rangle$ is an eigenvector of $\hat{H}$ with eigenvalue $E_n + \hbar\omega$. [ 4 ]

(c) Using the relations

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad \text{and} \quad a|n\rangle = \sqrt{n}|n-1\rangle,$$

calculate explicitly the expectation values $\langle\hat{p}\rangle$, $\langle\hat{p}^2\rangle$, $\langle\hat{x}\rangle$ and $\langle\hat{x}^2\rangle$ for a simple harmonic oscillator in the state $|n\rangle$. Obtain an expression for the product $\Delta x\Delta p$ as a function of $n$ (where $\Delta x = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2}$ etc), and confirm that it complies with Heisenberg's uncertainty relation. [ 8 ]

(d) Now consider the eigenvectors $|n\rangle$ and $|m\rangle$ where $n \neq m$. Find the values of $n$ and $m$ for which the inner product $\langle n|\hat{x}^2|m\rangle$ vanishes and the values for which this inner product does not vanish (show your working). [ 4 ]

**TURN OVER**

**B2.** Consider the following relations for angular momentum operators:

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y \qquad \hat{L}_- = \hat{L}_x - i\hat{L}_y \ ,$$

$$\hat{L}_+|lm\rangle = \hbar\sqrt{(l-m)(l+m+1)}|l,m{+}1\rangle \quad \hat{L}_-|lm\rangle = \hbar\sqrt{(l+m)(l-m+1)}|l,m{-}1\rangle \ ,$$

$$\hat{L}_z|lm\rangle = m\hbar|lm\rangle \qquad \hat{L}^2|lm\rangle = l(l+1)\hbar^2|lm\rangle \ .$$

For a system with orbital angular momentum $l = 1$ the eigenvectors $|lm\rangle$ can be labelled by the eigenvalue $m$ alone, and so one has three eigenvectors $|1\rangle_z$, $|0\rangle_z$, and $|-1\rangle_z$ for $m = 1, 0, -1$ respectively.

(a) Using the above relations for the angular momentum operators, show that the matrix representation of $\hat{L}_x$ for $l = 1$ in the basis of eigenvectors of $\hat{L}_z$ and $\hat{L}^2$ is given by

$$\hat{L}_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ .$$ [ 7 ]

(b) Now find the eigenvalues and (normalised) column vector representations of the eigenvectors of $\hat{L}_x$, neglecting global phase factors. [ 10 ]

(c) Calculate the probability that a measurement of $\hat{L}_x$ will give zero for a system that is in the state

$$|\psi\rangle = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \ .$$ [ 3 ]

**B3.** (a) State the four postulates of quantum mechanics. For concreteness, consider a one-dimensional system e.g. a particle moving along the $x$-axis. [ 6 ]

(b) An operator $\hat{\Omega}$ in an $n$ dimensional Hilbert space has eigenvectors $|\omega_i\rangle$ with corresponding eigenvalues $\omega_i$, where $i$ is an integer from 1 to $n$. The expectation value of $\hat{\Omega}$ is denoted by $\langle\hat{\Omega}\rangle$. Starting from the expression

$$\langle\hat{\Omega}\rangle = \sum_{i=1}^{n} P(\omega_i)\,\omega_i\,,$$

where $P(\omega_i)$ is the probability of obtaining the eigenvalue $\omega_i$ from a measurement of $\hat{\Omega}$, show that $\langle\hat{\Omega}\rangle$ can be written as

$$\langle\hat{\Omega}\rangle = \langle\psi|\hat{\Omega}|\psi\rangle\,,$$

where $|\psi\rangle$ is the state vector for the system. [ 5 ]

(c) Explain what is meant by an "eigenbasis" of a Hilbert space. Now consider the case of two of the eigenvalues of the operator $\hat{\Omega}$ in (b) above being degenerate i.e. $\omega_1 = \omega_2 = \omega$. Show that there are an infinite number of eigenbases for this Hilbert space. [ 4 ]

(d) Now consider another operator $\hat{\Lambda}$ with eigenvectors $|\lambda_i\rangle$, and corresponding eigenvalues $\lambda_i$ with no degeneracy. Suppose that $\hat{\Lambda}$ commutes with $\hat{\Omega}$ i.e. $[\hat{\Lambda},\hat{\Omega}] = 0$. Show that $\{|\lambda_i\rangle\}$ are also eigenvectors of $\hat{\Omega}$. [ 5 ]

**TURN OVER**

**B4.** (a) Write down an expression for the most general state ($|\psi\rangle_{AB}$) in a Hilbert space $V_A \otimes V_B$, where $V_A$ and $V_B$ are Hilbert spaces with orthonormal bases $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ respectively. Then give the condition for this general state to be separable and the condition for this state to be entangled. [ 4 ]

(b) Suppose that Alice has a single qubit state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ which she wishes to teleport to her friend Bob. To do this she creates an entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

keeping the first (left) qubit and sending the second (right) qubit to Bob. Show that twice the product state $|\phi\rangle|\psi\rangle$ can be written as

$$\begin{aligned}2|\phi\rangle|\psi\rangle = |B_0\rangle(\alpha|0\rangle + \beta|1\rangle) + |B_1\rangle(\alpha|1\rangle + \beta|0\rangle) \\ +|B_2\rangle(\alpha|0\rangle - \beta|1\rangle) + |B_3\rangle(\alpha|1\rangle - \beta|0\rangle),\end{aligned}$$

where

$$|B_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |B_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$

$$|B_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \qquad |B_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$ [ 6 ]

(c) Now show that the operators

$$\hat{I} = |0\rangle\langle 0| + |1\rangle\langle 1|, \qquad \hat{X} = |0\rangle\langle 1| + |1\rangle\langle 0|,$$
$$\hat{Y} = |0\rangle\langle 1| - |1\rangle\langle 0|, \qquad \hat{Z} = |0\rangle\langle 0| - |1\rangle\langle 1|,$$

can be used to transform the single qubit parts of the product state above into $|\phi\rangle$. That is, show that

$$\hat{I}(\alpha|0\rangle + \beta|1\rangle) = |\phi\rangle, \qquad \hat{X}(\alpha|1\rangle + \beta|0\rangle) = |\phi\rangle,$$

$$\hat{Y}(\alpha|1\rangle - \beta|0\rangle) = |\phi\rangle, \qquad \hat{Z}(\alpha|0\rangle - \beta|1\rangle) = |\phi\rangle.$$ [ 4 ]

 

(d) Explain step by step how Alice and Bob can make use of the above results to devise a scheme for teleporting the original qubit $|\phi\rangle$ without either of them knowing its state. [ 6 ]

**END OF PAPER**