UNIVERSITY OF SOUTHAMPTON PHYS6003W1

SEMESTER 1 EXAMINATION 2012/13

ADVANCED QUANTUM PHYSICS

Duration: 120 MINS

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*Answer **all** questions in **Section A** and two **and only two** questions in **Section B**.*

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it. Section B carries 2/3 of the total marks for the exam paper and you should aim to spend about 80 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

Only university approved calculators may be used.

Number of Pages 7

Section A

A1. Give the mathematical condition for a set of vectors $|V_1\rangle$, $|V_2\rangle$,....$|V_n\rangle$ to be *linearly independent*. [1]

Determine whether the following vectors are linearly independent or not (show your working):

(a)

$$|V_1\rangle = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad |V_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |V_3\rangle = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$$ [2]

(b)

$$|V_1\rangle = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \qquad |V_2\rangle = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix} \qquad |V_3\rangle = \begin{pmatrix} 0 \\ 1 \\ \sqrt{3} \end{pmatrix}$$ [2]

A2. Suppose that the vectors $|0\rangle, |1\rangle, |2\rangle$ form an orthonormal basis for a vector space. Determine whether the following are Hermitian operators on the space (show your working):

(a)

$$|0\rangle\langle 1| - i|1\rangle\langle 0|$$ [1]

(b)

$$|0\rangle\langle 0| + i|1\rangle\langle 0| - i|0\rangle\langle 1| + |2\rangle\langle 2|$$ [1]

A3. Show that Hermitian operators have real eigenvalues. [2]

Then show that eigenvectors which correspond to distinct eigenvalues of a Hermitian operator are orthogonal. [2]

A4. Assuming that the state $|\psi(t)\rangle$ is normalised, (i.e. $\langle\psi(t)|\psi(t)\rangle = 1$) find an expression for the time derivative, $\frac{d\langle\hat{Q}\rangle}{dt}$, of the expectation value of some time-independent operator $\hat{Q}$. [3]

Comment on the physical significance of the case when $\hat{Q} = \hat{H}$, where $\hat{H}$ is the time-independent Hamiltonian. [1]

A5. Suppose that the Hamilitonian of quantum system, $\hat{H}$, does not depend explicitly on time (t). Its eigenvalues E_i corresponding to the orthonormal eigenstates $|E_i\rangle$ are thus time-independent: $\hat{H}|E_i\rangle = E_i|E_i\rangle$. A generic state $|\psi(t)\rangle$ can be expanded using the eigenbasis of the Hamiltonian:

$$|\psi(t)\rangle = \sum_i a_i(t)|E_i\rangle \text{ with } (i = 1, 2, ...n)\,.$$

Find the explicit form of the coefficients $a_i(t)$ in terms of $a_i(0)$, and hence write down an expression for the time evolution of $|\psi(t)\rangle$. [3]

If at time $t = 0$ the generic state coincides with one of the eigenstates of the Hamiltonian, $|\psi(0)\rangle = |E_j\rangle$, then show that the subsequent time evolution is given by $|\psi(t)\rangle = \exp(-iE_jt/\hbar)|E_j\rangle$. Comment on the physical significance of the phase $\exp(-iE_jt/\hbar)$ if $|\psi(0)\rangle = |E_j\rangle$. [2]

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Section B

B1. A simple harmonic oscillator has a Hamiltonian, $\hat{H} = \frac{1}{2}(\hat{X}^2 + \hat{P}^2)\hbar\omega$, where $\hat{X}$ and $\hat{P}$ are position and momentum operators that satisfy $[\hat{X}, \hat{P}] = i$. Define the operators $a = (\hat{X} + i\hat{P})/\sqrt{2}$ and $a^\dagger = (\hat{X} - i\hat{P})/\sqrt{2}$.

(a) Show that $[a, a^\dagger] = 1$ and $\hat{H} = (a^\dagger a + \frac{1}{2})\hbar\omega$. [6]

(b) Defining $N = a^\dagger a$, and letting $|n\rangle$ be a normalised eigenvector of N with eigenvalues n, show that $n \geq 0$. [3]

(c) Then show that $Na^\dagger|n\rangle = (n+1)a^\dagger|n\rangle$. Explain why $a^\dagger|n\rangle$ cannot equal zero. [6]

(d) Now show that $Na|n\rangle = (n-1)a|n\rangle$. Explain why n must be a positive integer or zero. [5]

B2. Hermitian angular momentum operators J_i (for $i = 1, 2, 3$, and $\hbar = 1$) obey the commutation relations by $[J_i, J_j] = i\epsilon_{ijk}J_k$, where ϵ_{ijk} changes sign under interchange of any two indices and $\epsilon_{123} = 1$. The ladder operators J_+ and J_- are defined as $J_+ = J_1 + iJ_2$ and $J_- = J_1 - iJ_2$.

(a) Show that $J_+ = [J_3, J_+]$ and $J_- = -[J_3, J_-]$. [4]

Now consider the normalised state of angular momentum $|j, m\rangle$, where

$$J^2|j, m\rangle = j(j+1)|j, m\rangle$$

and

$$J_3|j, m\rangle = m|j, m\rangle.$$

(b) By evaluating the expectation values $\langle J_3\rangle$ and $\langle J_3^2\rangle$ show that $\Delta J_3 = 0$, where $\Delta J_3 = \sqrt{\langle J_3^2\rangle - \langle J_3\rangle^2}$. [5]

(c) Evaluate $\langle J^2\rangle$. [1]

(d) Using the commutation relations given in (a), as well as the commutator $[J^2, J_\pm] = 0$, show that

$$J_+|j, m\rangle = N_+|j, m+1\rangle \quad [3]$$

and

$$J_-|j, m\rangle = N_-|j, m-1\rangle, \quad [3]$$

where N_+ and N_- are normalisation constants (which do not need to be explicitly determined).

(e) Show that $\langle J_1\rangle = \langle J_2\rangle = 0$. [4]

B3. The operator $\hat{\mathbf{S}}$, which represents the spin of a spin-$\frac{1}{2}$ particle (such as the electron) may be written as $\hat{\mathbf{S}}=(\hat{S}_x, \hat{S}_y, \hat{S}_z)$ where

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are matrix representations of the (x, y, z) components of the spin operator.

(a) Find the eigenvalues (s_x^{up} and s_x^{down}) and normalised eigenvectors ($|\uparrow\rangle$ and $|\downarrow\rangle$) of the operator $\hat{S}_x$ i.e.

$$\hat{S}_x|\uparrow\rangle = s_x^{\mathrm{up}}|\uparrow\rangle, \quad \hat{S}_x|\downarrow\rangle = s_x^{\mathrm{down}}|\downarrow\rangle. \qquad [5]$$

Neglect phase factors.

(b) Compute the expectation value of the y and z-component of the particle's spin if it is in the state $|\uparrow\rangle$. [6]

(c) If the particle is in the state

$$|\psi\rangle = a\begin{pmatrix} 5 \\ 12 \end{pmatrix}$$

determine the normalisation factor a, where a is real and positive. [1]

What is the probability to find the particle with spin-up along the x-axis? [4]

(d) Suppose that after a measurement of the $\hat{S}_x$ operator on the state $|\psi\rangle$ the particle is found with spin up (along the x-axis). What is the probability to find the electron with spin down along the z-axis? [4]

B4. A simple harmonic oscillator in one dimension is defined by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2\,,$$

where $[\hat{x}, \hat{p}] = i\hbar$.

The raising and lowering operators are given respectively by

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}\,,$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}\,.$$

(a) Show that the ground state ($n = 0$) wavefunction is given by

$$\psi_0(x) = ce^{-m\omega x^2/2\hbar}\,.$$ [6]

(b) If $\psi_0(x)$ is normalised then show that (up to a phase)

$$c = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}.$$ [6]

$$\left[\text{note that } \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}\right]$$

(c) Then find explicit coordinate space wavefunctions (un-normalised) for the first two excited states (i.e. $n = 1$ and $n = 2$). [8]

END OF PAPER