SEMESTER 1 EXAMINATION 2013-2014

COSMOLOGY AND THE EARLY UNIVERSE

Duration: 120 MINS (2 hours)

This paper contains 8 questions.

Answer **all** questions in **Section A** and **one** question in **each** of **Section B** and **Section C**.

Each section carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on each.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

The default system of units is SI. You may give expressions in natural units, but you should state when you start to use them and note when you change unit systems. Throughout the paper the scale factor is normalized in such a way that at the present time $a_0 = 1$. The Friedmann equation is

$$H^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{k c^2}{a^2 R_0^2}\,.$$

Section A

A1. Consider Friedmann cosmological models. Write down the cosmological fluid equation in any equivalent form in terms of the total energy density ε, the pressure p and the scale factor $a(t)$. Specialise it to the case of a fluid corresponding to a cosmological constant. Show how the energy density depends on the scale factor in this particular case. [6]

A2. The physical size of the visible component of our Galaxy at the present time is about $25\,\text{kpc}$. Consider a past time $t_\star$ such that $a(t_\star) = 2/3$. What was the approximate physical size of the of the visible component of our Galaxy at $t_\star$ neglecting any astrophysical evolution? What was the co-moving size of the of the visible component of our Galaxy at $t_\star$? [2]

A3. List the three main features of the Cosmic Microwave Background radiation and their physical origin (no more than 50 words each). [6]

A4. Derive an expression for the baryon-to-photon number density ratio at the present time $\eta_{B,0} \equiv n_{B,0}/n_{\gamma,0}$ in terms of the quantity $\Omega_{B,0}\,h^2$, where $\Omega_{B,0}$ is the baryon energy density parameter and h is the Hubble constant in units of $100\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$. For the critical energy density at the present time use the value $\varepsilon_{c,0} = 10\,\text{GeV}\,\text{m}^{-3}$, for the mass of nucleons the approximate value $m_N\,c^2 = 1\,\text{GeV}$, for the number density of photons at the present time $n_{\gamma,0} = 400\,\text{cm}^{-3}$ and for the megaparsec the value $1\,\text{Mpc} = 3 \times 10^{16}\,\text{m}$. [6]

Section B

B1. Consider Friedmann cosmological models.

(a) What is meant by Friedmann cosmological models? [2]

(b) (i) What are the possible space geometries and what is the assumed number of space dimensions?

(ii) What is the geometrical parameter that discriminates between the possible space geometries and what are its respective values?

(iii) The physical parameter that determines the geometry of the Universe is the energy density parameter Ω. In terms of what two quantities is it defined and how?

(iv) What is the value, or range of values, of Ω for each possible space geometry? [5]

(c) Does the geometry of the Universe unambiguously determine the fate of the expansion of the Universe? Motivate your answer qualitatively, without any mathematical derivation. [3]

(d) Write down the Friedmann-Robertson-Walker metric in terms of the scale factor $a(t)$ and of the comoving spatial metric $d\ell^2_{(0)}$. [4]

(e) Starting from the spatial metric $d\ell^2$, derive how the comoving size of a bound system scales with the scale factor $a(t)$. [3]

(f) The proper distance of a luminous source at the present time is defined as

$$d_{\mathrm{pr}}(t_0) = \int_C d\ell_{(0)}\,,$$

where the integral is taken along the geodesic C between us and the source. Suppose that a light signal is emitted at the time t_{em} by the source. How can the proper distance $d_{\mathrm{pr}}(t_0)$ be calculated knowing the time dependence of the scale factor $a(t)$? [3]

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B2. Consider Friedmann cosmological models.

(a) How is the proper distance of an astronomical object defined? Starting from the Friedmann-Robertson-Walker metric, show that this can be mathematically expressed as

$$d_{\rm pr}(t) = a(t)\, R_0 \int_0^r \frac{dr'}{\sqrt{1 - k\, r'^2}}\,,$$

specifying the physical meaning of R_0, r and k. Is the proper distance a physical distance or a co-moving distance? [5]

(b) Starting from the expression for the proper distance of an astronomical object $d_{\rm pr}(t)$, derive the theoretical Hubble's law. How is the expansion rate defined? What is Hubble's constant? [2]

(c) Consider a luminous signal emitted by an astronomical source at a past time $t_{\rm em}$. Derive an expression for the proper distance of the source at the present time t_0 as a function of $t_{\rm em}$. [4]

(d) How is the redshift z of a luminous source emitting a signal at $t_{\rm em}$ defined? Derive an expression for the redshift in terms of the scale factor $a_{\rm em} \equiv a(t_{\rm em})$ at the time of the emission. How is the lookback time defined? Derive an approximate expression for the lookback time in terms of the redshift z to first order in z. Using $H_0^{-1} = 14\,{\rm Gyr}$, what is the distance of an object with $z = 0.1$? [6]

(e) Consider now a flat matter dominated Universe. Using the same value of H_0 as in part (d), calculate how long after the Big Bang the light of the most distant observed Galaxy, which has a redshift $z = 8.6$, was emitted.
(Hint: In a flat matter dominated Universe $t_0 = 2\, H_0^{-1}/3$.) [3]

Section C

C1. (a) Discuss qualitatively the main features and phenomenological motivation for Big Bang Nucleosynthesis (BBN) indicating in particular:

(i) the dominant primordial nuclear abundances in the Universe and their approximate values;

(ii) the main differences with stellar nucleosynthesis;

(iii) the importance of Deuterium synthesis. [5]

(b) Consider the approximate expression below for the expansion rate as a function of temperature T, which holds in the radiation-dominated regime:

$$H(T) \simeq 0.21\ \sqrt{g_R(T)}\ \left(\frac{k_B\,T}{\mathrm{MeV}}\right)^2\ s^{-1}\,,$$

where $g_R(T)$ is the number of ultra-relativistic degrees of freedom. Derive an approximate expression for time as a function of temperature. [4]

(c) The approximate value for the temperature of the coupled particle species (photons, electrons and nucleons) at the time of nucleosynthesis is $k_B\,T_{\rm nuc} \simeq 0.065\,\mathrm{MeV}$. Considering that at this time (after electron-positron annihilations) the temperature of neutrinos is related to the temperature of the coupled particles T by

$$T_\nu = \left(\frac{4}{11}\right)^{\frac{4}{3}}\ T \quad ,$$

show that $g_R(T \simeq T_{\rm nuc}) \simeq 3.36$ (Hint: Take into account a factor $7/8$ in the calculation of the contribution to g_R from fermionic species.). [5]

(d) Sketch the dependence on time t of the neutron-to-proton number density ratio, (n_n/n_p), in a logarithmic plot in the interval of time corresponding to temperatures $10\,\mathrm{MeV} \geq T \geq T_{\rm nuc} \simeq 0.065\,\mathrm{MeV}$. Indicate on the plot the main stages of the (n_n/n_p) evolution and the approximate analytical

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functional dependence of (n_n/n_p) either on temperature or on time (as appropriate). Indicate the approximate values of those special times, and of the corresponding approximate values of temperatures, that separate the different stages, explaining very briefly their physical meaning in a separate legend. [6]

C2. (a) Within the 'old' Hot Big Bang Model (no inflationary stage) there are three basic fundamental problems. List them first and then discuss them briefly (no more than 70 words for each problem, no more than 150 words in total). [5]

(b) Consider Friedmann cosmological flat models with a singularity, such that $a(t=0)=0$. Starting from the Friedmann-Robertson-Walker metric, show that the horizon distance in the co-moving scale, at a generic time t, is given by

$$d_H^{(0)}(a) = \int_0^a \frac{c\, da'}{H(a')\, a'^2},$$

where a is the scale factor at the time t. [5]

(c) Calculate $d_H^{(0)}(a)$ for the case of a matter dominated Universe, solving the integral explicitly. Discuss briefly (no more than 150 words) why the result is regarded as a problem (the horizon problem) in light of the cosmological observations, considering, in particular, the properties of the cosmic microwave background radiation. [3]

(d) Consider now the case where the matter dominated regime is preceded by an inflationary stage that starts at $t = t_{\rm i}$ and ends at $t = t_{\rm f}$ with an expansion rate H_i.

(i) Write down an expression for $a(t)$, the scale factor time evolution law, during the inflationary stage.

(ii) How is the number of e-folds N defined?

(iii) Show that the horizon distance at the present time, neglecting the period before $t_{\rm i}$ and for $a_{\rm f} \equiv a(t_{\rm f}) \gg a_{\rm i} \equiv a(t_{\rm i})$, is given approximately by

$$d_{H,0} \simeq c\, H_0^{-1} \left(\sqrt{a_{\rm f}}\, e^N + 2 \right),$$

where $a_{\rm f}$ is the value of the scale factor at the end of the inflationary

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period and N is the number of e-folds.

(iv) Why does this expression show that the occurrence of an inflationary stage can solve the horizon problem? [7]

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