SEMESTER 1 EXAMINATION 2014-2015

COSMOLOGY AND THE EARLY UNIVERSE

Duration: 120 MINS (2 hours)

This paper contains 8 questions.

Answer **all** questions in **Section A** and **one** question in **each** of **Section B** and **Section C**.

Each section carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on each.

A Sheet of Physical Constants is provided with this examination paper.

Only university approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

The default system of units is SI. You may give expressions in natural units, but you should state when you start to use them and note when you change unit systems. Throughout the paper the scale factor is normalized in such a way that at the present time $a_0 = 1$. The Friedmann equation is

$$H^2 = \frac{8\,\pi\,G}{3\,c^2}\,\varepsilon - \frac{k\,c^2}{a^2\,R_0^2}\,.$$

Section A

A1. Define and write down the expression of the Hubble distance at the present time. Give a numerical estimate (1 significant figure) using $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

[5]

(seen problem) The Hubble distance at the present time is defined as that distance where the recession cosmological velocity equals the speed of light.

[1]

Recession cosmological velocities are described by the Hubble's law

$$v_{\rm pr} = H\, d_{\rm pr}\,,$$

where $H \equiv \dot{a}/a$ is the expansion rate and its value at present is given by the Hubble constant.

[1.5]

Therefore, the proper distance at present $d_{\rm pr}(t_0)$, corresponding to recession velocities equal to the speed of light (the Hubble distance or radius at the present time usually indicated with $R_{H,0}$), is given by

$$R_{\rm H,0} = c\, H_0^{-1}$$

[1]

In numbers

$$R_{\rm H}(t_0) \simeq \frac{3\times 10^5\,\mathrm{km\,s^{-1}}}{70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}} \simeq 4\,\mathrm{Gpc}\,.$$

[1.5]

A2. Consider an empty Universe within Friedmann cosmological models. Is it an open, flat or closed Universe? Motivate your answer. Using H_0 =

$70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and using $1\,\mathrm{Mpc} = 3 \times 10^{19}\,\mathrm{Km}$, calculate the age of the Universe, properly defined, in years (2 significant figures). [6]

(seen problem) From the Friedmann equation it can be immediately seen that for an empty Universe ($\varepsilon = 0$) one necessarily has $k = -1$, meaning that the empty Universe is open.

[2]

From the Friedmann equation one has $H^2 = \mathrm{const}/a^2 = H_0^2/a^2$

[2]

Therefore, one immediately has, since $H \equiv \dot{a}/a$, that $\dot{a} = H_0$, and therefore, $a(t) = H_0\, t$, having defined the origin of time as that particular time when $a = 0$. With this definition one has $t_0 = H_0^{-1} = (3/7) \times 10^{18}\,\mathrm{s} \simeq 14 \times 10^9\,\mathrm{yr}.$

[2]

A3. Consider an inflationary stage occurring in the time interval $[t_\mathrm{i}, t_\mathrm{f}]$, with $t_\mathrm{f} - t_\mathrm{i} = 10^{-32}\,\mathrm{s}$ and number of e-folds $N = 100$. What is the value of the expansion rate H_i during inflation (in s^{-1})? [5]

(seen problem) During inflation the scale factor experiences a (de Sitter) exponential expansion

$$a(t) = a_\mathrm{i}\, e^{H_\mathrm{i}\,(t - t_\mathrm{i})}\,,$$

[2]

(unseen) The number of e-folds N is defined as

$$N \equiv \ln \frac{a_f}{a_i}$$

and is therefore related to $\Delta t = t_\mathrm{f} - t_\mathrm{i}$ simply by

$$N = H_i\, \Delta t\,.$$

[2]

(unseen) Therefore, one has simply $H_i = N/\Delta t = 10^{34}\,\mathrm{s}^{-1}$

[1].

A4. Explain what is meant by Big Bang Nucleosynthesis (1 sentence). Consider $T_{\rm in}$, the initial (and highest) temperature during the expansion of the Universe within the ΛCDM model. The value of $k_B\,T_{\rm in}$ has to be assumed much greater than a certain value in order to reproduce correctly the measured primordial abundances. What is this value (1 significant figure) and what is its physical significance? [4]

(bookwork)

With BBN it is meant that early Universe stage during which the primordial nuclear abundances where synthesised.

[1]

(seen problem) In order to explain the observed primordial abundances the initial temperature of the Universe has to be much greater than $\sim 1\,\mathrm{MeV}$*.*

[1]

(seen problem) Below $k_B\,T_{\rm fr} \sim 1\,\mathrm{MeV}$*, neutrons-protons interconversion reactions freeze out and, therefore, only above this temperature one can assume a thermal equilibrium initial value for the neutron-to-proton abundance ratio a necessary ingredient not to spoil successful agreement between BBN predictions and measured primordial nuclear abundances.*

[2]

Section B

B1. (a) Consider the Friedmann equation. What is the name and the physical meaning of a, H, ε and k (no more than 1 sentence for each quantity)? [4]

(bookwork) The scale factor a describes the dynamics of the Friedmann-Robertson-Walker metric determining how proper distances scale during the expansion (Hubble's law).

[1]

H is the expansion rate and in terms of a is given by $H = \dot{a}/a$.

[1]

ε is the total energy density

[1]

k is the curvature parameter and determines the geometry of the Friedmann Universes.

[1]

(b) What values can k take and to what kinds of Universe do they respectively correspond? Consider a Friedmann cosmological model with an admixture of radiation, matter and non-vanishing cosmological constant. Is the value of k unambiguously determining the fate of the Universe expansion? Motivate your answer. [5]

(bookwork) k can take three values: $k = -1, 0, +1$.

[1.5]

(bookwork) They correspond respectively to an open, flat and closed Universe.

[1.5]

(seen problem) No, in the presence of a positive non-vanishing cosmological constant the value of the curvature parameter is not sufficient to determine the fate of the expansion of the Universe.

[1]

(seen problem) The reason is that in the presence of a positive cosmological constant, giving a repulsive gravitational effect, even though space can be closed, the expansion of the Universe can last indefinitely.

[1]

(c) What is the physical meaning of the critical energy density $\varepsilon_{\rm c}$?

(bookwork) It is the energy density corresponding to a flat Universe ($k = 0$).

[1.5]

Derive an expression for ε_c starting from the Friedmann equation.

(bookwork) Starting from the (given) Friedmann equation

$$H^2 = \frac{8\,\pi\,G}{3\,c^2}\,\varepsilon - \frac{k\,c^2}{a^2\,R_0^2}\,.$$

one can immediately see that for $k = 0$ the total energy density has to be given by

$$\varepsilon(t)|_{k=0} = \varepsilon_{\rm c}(t) \equiv \frac{3\,c^2\,H^2(t)}{8\,\pi\,G}\,.$$

[1.5]

How is the energy density parameter Ω defined?

(bookwork) The energy density parameter is defined as

$$\Omega = \frac{\varepsilon}{\varepsilon_{\rm c}}\,.$$

[1]

[4]

(d) Show that the Friedmann equation can be recast in terms of Ω_0 and H_0 as

$$\dot{a}^2(t) = H_0^2\,\Omega_0\,a^2(t)\,\frac{\varepsilon(t)}{\varepsilon_0} + H_0^2\,(1-\Omega_0)\,.$$ [7]

(seen problem) From the definition of Ω *and of* ε_c *we can recast the Friedmann equation as*

$$H^2 = H^2\,\Omega - \frac{k\,c^2}{a^2\,R_0^2}$$

[1.5]

(seen problem) If we now express this equation at the present time, we find

$$H_0^2 = H_0^2\,\Omega_0 - \frac{k\,c^2}{R_0^2}\,,$$

(where we used $a_0 = 1$*) and therefore the curvature term can be expressed in terms of* H_0 *and* Ω_0 *as*

$$-\frac{k\,c^2}{R_0^2} = H_0^2\,(1-\Omega_0)\,.$$

[2]

(seen problem) Inserting the expression for $k\,c^2/R_0^2$ *one finds*

$$H^2 = \frac{8\,\pi\,G}{3\,c^2}\,\varepsilon + \frac{H_0^2\,(1-\Omega_0)}{a^2}$$

and multiplying all terms by a^2

$$\dot{a}^2 = \frac{8\,\pi\,G}{3\,c^2}\,\varepsilon\,a^2 + H_0^2\,(1-\Omega_0)\,.$$

[1.5]

(seen problem) We can then express $8\,\pi\,G/(3\,c^2) = H_0^2/\varepsilon_{c,0}$ *and write*

$$\dot{a}^2(t) = H_0^2\,a^2(t)\,\frac{\varepsilon}{\varepsilon_{c,0}} + H_0^2\,(1-\Omega_0)\,.$$

TURN OVER

and finally, writing $\varepsilon_{\mathrm{c},0} = \Omega_0/\varepsilon_0$, *we find*

$$\dot{a}^2(t) = H_0^2\,\Omega_0\,a^2(t)\,\frac{\varepsilon}{\varepsilon_0} + H_0^2\,(1 - \Omega_0)\,.$$

[2]

B2. (a) Write down the cosmological fluid equation in any equivalent form you prefer. Show that for a fluid with equation of state $p = w\,\varepsilon$, where $w = \text{const}$, the total energy density depends on the scale factor a as

$$\varepsilon(a) = \frac{\varepsilon_0}{a^{3\,(1+w)}}\,.$$

[4]

(bookwork) The cosmological fluid equation can be written as

$$\frac{d(\varepsilon\, a^3)}{dt} = -p\,\frac{d\,a^3}{dt}\,,$$

or equivalently as

$$\dot{\varepsilon} = -3\,\frac{\dot{a}}{a}\,(\varepsilon + p)\,.$$

[2]

(seen problem) In the case of a fluid with $p = w\,\varepsilon$, from the second form, one obtains

$$\frac{\dot{\varepsilon}}{\varepsilon} = -3\,(1+w)\,\frac{\dot{a}}{a}\,,$$

that, considering that $\dot{X}/X = d\ln X/dt$, immediately yields

$$\varepsilon(a) = \frac{\varepsilon_0}{a^{3\,(1+w)}}\,.$$

[2]

(b) Derive the acceleration equation combining the Friedmann equation with the fluid equation. [5]

(seen problem)

Starting from the Friedmann equation (given on the cover page) and *multiplying both RH and LH sides by* a^2*, one obtains*

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\varepsilon a^2 - \frac{kc^2}{R_0^2}.$$

[1]

(seen) At this point, differentiating with respect to time, the curvature term cancels out and

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\left[\dot{\varepsilon} a^2 + 2\dot{a} a \varepsilon\right].$$

[1]

Factorising a term εa^2 *one then obtains*

$$2\dot{a}\ddot{a} = \frac{8\pi G}{3c^2}\varepsilon a^2\left[\frac{\dot{\varepsilon}}{\varepsilon} + 2\frac{\dot{a}}{a}\right].$$

[1]

Using the fluid equation one can now plug into the last equation $\dot{\varepsilon}/\varepsilon = -3\,[(\varepsilon + p)/\varepsilon]\,(\dot{a}/a)$ *obtaining straightforwardly the acceleration equation*

$$\ddot{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3p)\,a.$$

[2]

(c) Consider now *flat* one-fluid Friedmann cosmological models with an equation of state $p = w\,\varepsilon$ and $w = \mathrm{const} > -1$.

- Define the age of the Universe t_0 and derive an expression in terms of w and H_0.
- Derive an expression for the scale factor $a(t)$, in terms of t_0, H_0 and w.
- What is that particular value of w such that $t_0 = H_0^{-1}$?
- Which is the other (non-flat) Friedmann cosmological model with $t_0 = H_0^{-1}$?

[11]

(seen problem) We can first of all plug the given expression for $\varepsilon(a)$in the case $p = w\,\varepsilon$ into the Friedmann equation for a flat Universe ($k = 0$), finding

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\,\pi\,G\,\varepsilon_0}{3\,c^2}}\,a^{-\frac{3}{2}\,(1+w)}\,,$$

where we have only considered the expansion solution.

[2]

From the Friedmann equation we can also replace ε_0 with H_0,

$$\varepsilon_0 = \frac{3\,H_0^2\,c^2}{8\,\pi\,G}\,,$$

(the critical energy density at the present time) in way that one can write

$$a^{\frac{1+3\,w}{2}}\,da = H_0\,dt\,.$$

[2]

Assuming that there is a time when $a = 0$, the age of the Universe is defined as the time elapsed from this time conventionally set as the origin of time. Integrating between 0 and t_0 one finds for the age of the Universe t_0

$$t_0 = \frac{2\,H_0^{-1}}{3\,(1+w)}\,,$$

that is indeed defined for $w > -1$.

[2]

Going back and integrating between 0 and a generic time t one then finds

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3\,(1+w)}}\,,$$

TURN OVER

that indeed for $t = 0$ *implies* $a(t) = 0$ *for* $w > -1$.

[2]

(seen problem) The particular value of w *implying* $t_0 = H_0^{-1}$ *is* $w = -1/3$.

[1]

(seen) The other simple model that gives $t_0 = H_0^{-1}$, *is the empty Universe for* $\varepsilon = 0$ *since one can see from the Friedmann equation that* $\dot{a} = H_0$, *implying* $t_0 = H_0^{-1}$.

[2]

Section C

C1. (a) What are the three main observational features of the Cosmic Microwave Background (CMB) radiation spectrum? [5]

(bookwork)

Spectrum. The spectrum of the CMB radiation is very close to a thermal equilibrium spectrum described by the Planckian spectrum with a temperature $T = 2.725\,K$*. The deviations* $\Delta\varepsilon/\varepsilon$ *in the intensity are smaller than* 10^{-4}*.*

[1]

Dipole. The temperature of the CMBR spectrum presents a dipole anisotropy $\Delta T/T \sim 10^{-3}$ *that is explained by the Doppler effect due to the motion of the Earth with respect to the comoving system with a velocity* $v \sim 370\,\mathrm{km\,s^{-1}}$ *that is the result of the composition of different velocities (Earth around the Sun, the solar system around the galactic center, the Galaxy in the Local Group, the infall of the Local Group toward the Hydra-Centaurus super-cluster).*

[1.5]

Anisotropies. After the dipole anisotropy is subtracted, the temperature fluctuations at higher multipoles are found to be much smaller, $\Delta T/T \sim 10^{-5}$*. (Experiments have a finite resolution and therefore they can measure the anisotropies only on angular scales larger than the resolution. If the correlation function is expanded in Legendre polynomials, they measure the multipole moments* C_l*.) Current observation measure* C_l *up to* $l \simeq 1400$ *and find that the* C_l*'s (the angular power spectrum) exhibit a series of peaks characterised by their position, at some value* $l^{(i)}_{\rm peak}$*, and by the height.*

[2.5]

(b) Determine (1 significant digit) the value of $k_B\,T^{RM}_{\rm eq}$, where $T^{RM}_{\rm eq}$ is the

TURN OVER

temperature at the matter-radiation equality time. Use for the radiation energy density parameter at the present time $\Omega_{R,0} \simeq 0.75 \times 10^{-4}$, for the matter energy density parameter $\Omega_{M,0} \simeq 0.32$, for the Boltzmann constant $k_B = 0.86 \times 10^{-4}\,\mathrm{K\,eV^{-1}}$.

[3]

(seen problem) The value of the scale factor at the matter-radiation equality time is simply given by

$$a_{\rm eq}^{RM} = \frac{\Omega_{R,0}}{\Omega_{M,0}} \simeq 2.34 \times 10^{-4}\,.$$

[1.5]

Considering that $T \propto 1/a$ *and that* $T_0 \simeq 2.725\,\mathrm{K}$*, one then finds*

$$k_B\,T_{\rm eq}^{RM} = \frac{k_B\,T_0}{a_{\rm eq}^{RM}} \simeq 1.0\,\mathrm{eV}\,.$$

[1.5]

(c) Considering the observational features of the CMB radiation and the properties of photons, explain why the distribution function of relic photons (giving the average occupation number of each quantum state) is very well approximated by the Bose-Einstein distribution,

$$f_{\gamma,0}(p) = \frac{1}{e^{\frac{c\,p}{k_B\,T_0}} - 1}\,.$$

[3]

(seen problem) Photons of the CMB radiation (relic photons) are experimentally found to be well described by a thermal equilibrium spectrum and, therefore, their distribution function can be well approximated by an equilibrium distribution function. [1]

Since photons are bosons, their equilibrium distribution is the Bose-Einstein distribution.

[1]

Since photons are massless, their energy is simply given by $E = p\,c$.

[1]

(d) Derive the number density of relic photons at the present time in terms of the CMB radiation temperature T_0. Calculate its numerical value with two significant figures using for the Boltzmann constant $k_B = 0.86 \times 10^{-4}\,\mathrm{K\,eV^{-1}}$.
Hint: You might find useful

$$\int_0^\infty dx\,\frac{x^2}{e^x - 1} = 2\,\zeta(3) \simeq 2.4\,.$$

[7]

(seen problem) The calculation can be more conveniently made in the Natural System ($c = k_B = \hbar = 1$). In this case for relic photons the number density can be calculated from the distribution function as

$$n_{\gamma,0} = g_\gamma \int \frac{d^3 p}{(2\,\pi)^3}\,\frac{1}{e^{p/T_0} - 1}\,.$$

where $g_\gamma = 2$ is the number of the spin degrees of freedom (or the spin degeneracy) of photons.

[2]

Integration over the solid angle gives then

$$g_\gamma \int \frac{d^3 p}{(2\,\pi)^3}\,\frac{1}{e^{p/T_0} - 1} = \frac{1}{\pi^2}\int_0^\infty dp\,\frac{p^2}{e^{p/T_0} - 1}\,.$$

[1]

Changing the variable of integration from p to $x \equiv p/T_0$, one finds

$$n_{\gamma,0} = \frac{2\,\zeta(3)}{\pi^2}\,T_0^3\,,$$

where we used the given integral

$$\int_0^\infty dx \, \frac{x^2}{e^x - 1} = 2\,\zeta(3)\,,$$

where $\zeta(x)$ is the ζ-Riemann function and $\zeta(3) \simeq 1.2$. Plugging back the fundamental constants in the International system one can write

$$n_{\gamma,0} = \frac{2\,\zeta(3)}{\pi^2} \frac{(k_B\,T_0)^3}{(\hbar\,c)^3}\,.$$

[2]

Using $k_B = 0.86 \times 10^{-4}\,\mathrm{K\,eV^{-1}}$ and $T_0 = 2.725\,K$, one then finds (2 significant figures)

$$n_{\gamma,0} = \frac{2\,\zeta(3)}{\pi^2} \frac{(k_B\,T_0)^3}{(\hbar\,c)^3} \simeq \frac{2.4}{\pi^2} \frac{(0.86 \times 10^{-4} \times 2.725)^3}{(197 \times 10^6 \times 10^{-13})^3} \mathrm{cm}^{-3} \simeq 410\,\mathrm{cm}^{-3}\,.$$

[2]

(e) Derive the distribution of the relic photons at the matter-radiation decoupling time $t_{\mathrm{dec}}^{\mathrm{RM}}$. Explain the physical meaning of the result (1 sentence). [2]

In the early Universe the momentum of a free particle like relic photons scales like $\vec{p}(t) = \vec{p}_0/a$.

[1]

(seen problem) Since particles are free, the occupation number of each quantum state does not change but only their associated momentum. Therefore, one has

$$f_\gamma(\vec{p}, t) = f_\gamma(\vec{p}_0, t_0) = f_\gamma(\vec{p}\,a, t_0) = \frac{1}{e^{\frac{c\,p\,a}{k_B\,T_0}} - 1},$$

that is still a Bose-Einstein distribution with $T(t) = T_0/a$ meaning that expansion preserves the thermal equilibrium distribution of photons

[1]

TURN OVER

C2. (a) What are the main features of the ΛCDM model? Specify in particular what is the matter-energy budget at present, distinguishing the contributions from ordinary baryonic matter and the contribution from Dark Matter. [5]

(seen problem) The ΛCDM model is a flat model ($\Omega_0 = 1$) with an admixture of a radiation component ($\Omega_{R0} \sim 10^{-4}$), a matter component and a cosmological constant-like fluid (a particular kind of dark energy) component.

[3]

Within the matter component only $1/6$ is explained by ordinary baryonic matter, while the remaining $5/6$ have to be ascribed to the presence of some mysterious form of Dark Matter.

[2]

(b) The cosmological redshifts of supernovae indicate the approximate relation

$$\Omega_{\Lambda,0} \simeq 1.5\,\Omega_{M,0} + 0.25\,.$$

Write down the complementary relation linking $\Omega_{M,0}$ to $\Omega_{\Lambda,0}$ coming from the study of the CMBR anisotropies. What is the specific feature in the CMB radiation anisotropies that strongly supports it? Combining the two relations, calculate the values of $\Omega_{\Lambda,0}$ and $\Omega_{M,0}$ (1 significant figure). [5]

(bookwork) From the position of the first peak in the CMBR acoustic peaks, we deduce that the Universe is flat and therefore $\Omega_0 = \Omega_{M,0} + \Omega_{\Lambda,0} = 1$. [2.5]

(seen problem) Inserting $\Omega_{\Lambda,0} = 1 - \Omega_{M,0}$ in the relation from the supernovae observations, one straightforwardly finds

$$2.5\,\Omega_{M,0} = 0.75 \Rightarrow \Omega_{M,0} \simeq 0.3 \Rightarrow \Omega_{\Lambda,0} = 0.7\,.$$

[2.5]

(c) Discuss qualitatively which astronomical observations support the existence of Dark Matter. What is the interpretation of the nature of Dark Matter more strongly supported by the current astronomical and cosmological observations? [5]

(bookwork) More directly we infer the existence of a Dark Matter from anomalous motions of stars in Galaxies and of Galaxies within clusters and superclusters of Galaxies: they are too fast to remain bound under the gravitational attraction of just baryonic matter. The Dark Matter plays therefore the role of a cosmic glue. [2.5]

(seen problem) The astronomical observations strongly favour the existence of a new elementary particle, stable on cosmological scales, that would constitute the Dark Matter.

[2.5]

(d) Most of the visible matter in Galaxies is concentrated within a few kpc from the centre. Show how the Newton's law of gravity implies that the speed of a star at a distance $R \gtrsim 10\,\mathrm{kpc}$, i.e. far from the central bulge, should decrease like $v(R) \propto 1/\sqrt{R}$ in the absence of Dark Matter. [5]

(seen problem) Star motions can be approximated as circular and therefore the absolute value of the radial acceleration of a star at distance R is related to the velocity simply by $a = v^2/R$.

[2]

(bookwork) The gravitational attraction from the mass M of the galactic central bulge gives an acceleration

$$a = \frac{G\,M}{R^2}$$

[2]

(seen problem) Imposing that gravitational attraction of the central bulge

TURN OVER

explains the radial acceleration one has therefore

$$v^2 \propto R^{-1} \Rightarrow v \propto 1/\sqrt{R}\,.$$

[1]

END OF PAPER