SEMESTER 2 EXAMINATIONS 2013-2014

MPHYS SYNOPTIC PHYSICS

DURATION 120 MINS (2 Hours)

This paper contains 9 questions

Answer **all** questions in **Section A**, **only one** question in **Section B,** and **only one** question in **Section C.**

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 minutes on it.

Section B carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 minutes on it.

Section C carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 minutes on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only University approved calculators may be used.

A foreign language translation dictionary (paper version) is permitted provided it contains no notes, additions or annotations.

Section A

A1. The average power radiated by an oscillating electric dipole is given in SI units by

$$\mathrm{W} = \frac{4\pi^3}{3}\frac{c}{\varepsilon_0}\frac{p_0^2}{\lambda^4},$$

where p_0 is the amplitude of the oscillating dipole, λ is the wavelength of the radiated field, ε_0 is the permittivity of free space and c is the speed of light. Show that this equation is dimensionally consistent. [4]

A2. The spin wave function of a spin-½ particle may be described in terms of the basis states

$$e_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad e_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which are eigenfunctions of the spin angular momentum operator S_z. Show that

$$S_x e_\uparrow = \frac{\hbar}{2} e_\downarrow$$

where

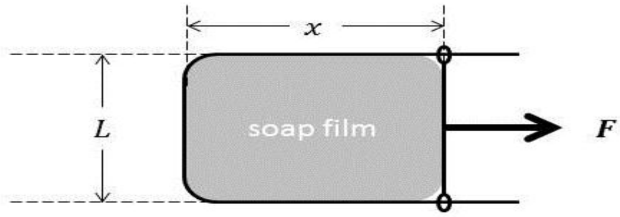
$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the eigenstate of S_x in which the particle spin points in the $+x$ direction, expressed as a superposition of $e_\uparrow$ and $e_\downarrow$. Hence show that if the particle spin is initially oriented along $+x$, there is a probability of $\frac{1}{2}$ that a subsequent measurement of S_z will yield the value $\frac{\hbar}{2}$. [4]

A3. According to the highway code, the typical stopping distance for a motor vehicle travelling at 70 mph is 75 m. (This is the distance travelled after the driver has hit the brakes.) Estimate the g-force experienced by the driver while the car is braking; i.e. the average deceleration in units of the acceleration due to gravity. Note that 1 mile = $^8/_5$ km. [4]

A4. Sketch a ray diagram that explains the operating principle of a hand magnifying lens. Explain the type of image (real, virtual, upright, inverted) observed by the user looking through the lens. State the condition on the distance between the lens and the object that must be satisfied for a magnified image to be formed. [4]

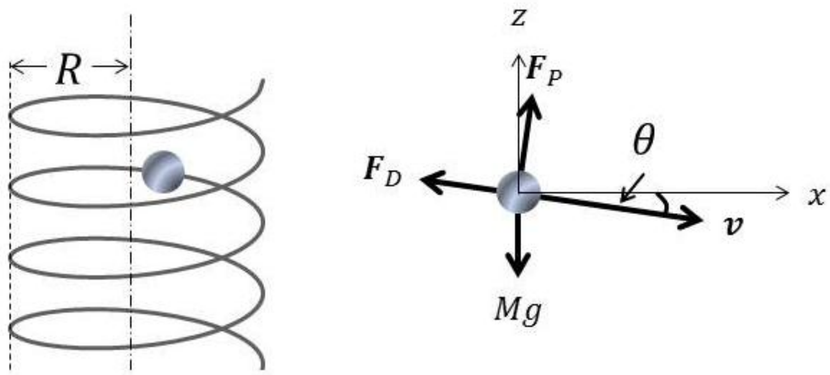
A5.



Consider a soap film stretched across a rigid wire framework closed by a sliding wire on the right, as shown in the diagram. In equilibrium a force $F = 2\gamma L$ is applied to the sliding wire to balance the force due to the surface tension γ. Write down the 1st & 2nd laws of thermodynamics as applied to an infinitesimal change dA in the total area $A = 2Lx$ of the top and bottom soap film surfaces. Hence prove that in an adiabatic stretching of the film (no heat transferred), the internal energy U satisfies $(\partial U/\partial A)_S = \gamma$. [4]

SECTION B

B1.



a) **b)**

A metal bead of mass M is free to slide on a rigid insulating wire in the form of a vertical helix of radius R, as shown in a) above. The wire makes a constant angle of θ to the horizontal plane. The forces acting on the bead in the vertical plane containing its velocity $\boldsymbol{v}$ are shown in b): $\boldsymbol{F}_D$ is the drag force due to friction between the bead and the wire, and $\boldsymbol{F}_P$ is one component of the reaction force that the wire exerts on the bead

(i) The bead slides down the helix and reaches a steady state of motion at constant speed v. Write down expressions for F_D and F_P, explaining your reasoning. [4]

(ii) The bead has angular momentum $M(\boldsymbol{r} \times \boldsymbol{v})_z$ about the z-axis. Explain why the reaction force that the wire exerts on the bead also contains a horizontal component $\boldsymbol{F}_H$ pointing in towards the axis of the helix. Write down an expression for F_H. [4]

(iii) The magnitude of the drag force F_D is proportional to the magnitude of the *total* reaction force exerted by the wire on the bead. The proportionality constant is α, the coefficient of friction. Show that

$$F_D = \alpha \sqrt{\left(\frac{M(v\ \cos\theta)^2}{R}\right)^2 + (Mg\cos\theta)^2}.$$

[4]

(iv) Show that the constant speed v at which the bead slides in a steady state is given by

$$v^2 = \frac{gR}{\cos\theta} \sqrt{\left[\left(\frac{\tan\theta}{\alpha}\right)^2 - 1\right]}.$$

[4]

(v) A charge Q is applied to the bead, and a uniform magnetic field B pointing vertically upwards is switched on. Explain why the steady-state speed v now depends on B. Discuss whether v increases or decreases in the presence of the field. [4]

B2. An insulating crystal contains N paramagnetic ions that have a doublet ground state. In an applied magnetic field B, the ground state splits into two levels with energies $E = \pm\mu B$. In thermal equilibrium at temperature T the magnetic degrees of freedom of this system are described by a canonical ensemble, for which the single-molecule partition function Z_1, the free energy F, the internal energy U, and the entropy S are related by

$$Z_1 = \sum_i \exp\left(-\frac{E_i}{k_B T}\right);$$

$$F = U - TS = -Nk_B T \ln Z_1;$$

$$U = N\frac{\sum_i E_i \exp\left(-\frac{E_i}{k_B T}\right)}{Z_1} = Nk_B T^2 \frac{\partial \ln Z_1}{\partial T},$$

where E_i is the energy of level i, and k_B is Boltzmann's constant.

(i) Show that the partition function of this magnetic system is given by

$$Z_1 = 2\cosh\left(\frac{\mu B}{k_B T}\right).$$

[4]

(ii) Show that the entropy is given by

$$\frac{S}{Nk_B} = -\frac{\mu B}{k_B T}\tanh\left(\frac{\mu B}{k_B T}\right) + \ln\left[2\cosh\left(\frac{\mu B}{k_B T}\right)\right].$$

[4]

(iii) Explain qualitatively why the entropy of the system decreases monotonically with increasing magnetic field at all temperatures. [4]

(iv) Find values for the entropy of the system a) in the limit as $B/T \to 0$, and b) in the limit as $B/T \to \infty$. [4]

(v) Suppose that the crystal is placed in an intense magnetic field B_i while it is cooled to a low temperature T_i. The magnetic field is then switched off abruptly, so that no heat flows in or out of the crystal while the field is changing (adiabatic demagnetisation). After the switch-off, there remains a small residual magnetic field $B_f \ll B_i$. Explain why the magnetic system cools down during the demagnetisation process, and write down an expression for the final temperature that it attains.

[4]

TURN OVER

SECTION C

C1. In a Lorentz boost to a new inertial frame moving with velocity $\boldsymbol{v}$ in the z-direction, electric and magnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ transform as

$$E_z^{/} = E_z \qquad B_z^{/} = B_z$$

$$\boldsymbol{E}_T^{/} = \gamma(\boldsymbol{E}_T + \boldsymbol{v} \times \boldsymbol{B}_T) \qquad \boldsymbol{B}_T^{/} = \gamma\left(\boldsymbol{B}_T - \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}_T\right)$$

where $= \sqrt{1-(v/c)^2}^{-1}$, and the subscripts z and T denote field components parallel and perpendicular to $\boldsymbol{v}$ respectively. You may find the following vector identity helpful for this question:

$$\boldsymbol{a}.(\boldsymbol{b} \times \boldsymbol{c}) = -\boldsymbol{c}.(\boldsymbol{b} \times \boldsymbol{a})$$

(i) Prove that the quantity $\boldsymbol{E}.\boldsymbol{B}$ remains *invariant* under the boost, so that $\boldsymbol{E}^{/}.\boldsymbol{B}^{/} = \boldsymbol{E}.\boldsymbol{B}$. [4]

(ii) Prove that $(c^2B^2 - E^2)$ also remains invariant. [4]

(iii) Suppose that these fields are produced by a set of point charges that are all at rest in one particular inertial frame. Explain why in all other inertial frames $\boldsymbol{E}^{/}$ and $\boldsymbol{B}^{/}$ must be mutually perpendicular. Sketch a diagram for the special case of a single point charge, showing the orientation of $\boldsymbol{v}$, $\boldsymbol{E}$, $\boldsymbol{E}^{/}$ and $\boldsymbol{B}^{/}$. [4]

(iv) Now consider the electromagnetic field associated with an linearly polarised infinite plane wave propagating along the z-direction in free space, represented in complex notation by

$$\boldsymbol{E} = \hat{\boldsymbol{x}} E_{\boldsymbol{x}} = \hat{\boldsymbol{x}} E_{\boldsymbol{0}} \exp[ik(z - ct)],$$

where E_0 is the electric field amplitude and k is the wavevector. With the help of the Maxwell equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

or otherwise, find the value of the invariant $(c^2B^2 - E^2)$ for this wave. [4]

(v) Show that the energy density

$$U = \frac{1}{2}\varepsilon_0(E^2 + c^2B^2)$$

associated with the electromagnetic wave described in (iv) is ***not*** invariant under a Lorentz boost to a frame moving with velocity $\boldsymbol{v}$ in the z-direction. Explain qualitatively why this result does not violate the principle of conservation of energy. [4]

TURN OVER

C2. Consider a 2-dimensional (2-D) “hydrogen atom” formed by a particle of mass m and charge $-e$ constrained to move only in the x-y plane and bound by the Coulomb interaction to a particle of charge $+e$ located at the origin. The time-independent Schrödinger equation for this system may be written as

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}\right)\psi(r,\phi)-\frac{e^2}{4\pi\varepsilon_0 r}\psi(r,\phi)=E\psi(r,\phi),$$

where (r,ϕ) are polar coordinates of the position of the particle, ψ is its wavefunction and E is its energy eigenvalue.

(i) The ground state wavefunction of the 2-D hydrogen atom is of the form

$$\psi = A\exp(-r/a_B^{2D})$$

where a_B^{2D} is the 2-D Bohr radius and A is a constant. Show that ψ satisfies the given Schrödinger equation, and find an expression for a_B^{2D} in terms of the constants $\hbar$, m, e and ε_0. [4]

(ii) Find an expression for E_1^{2D}, the energy eigenvalue of the 2-D hydrogen atom corresponding to the eigenfunction given in (i). [3]

(iii) For the 3-dimensional (3-D) hydrogen atom, the Bohr radius and ground state energy are given by

$$a_B^{3D}=\frac{4\pi\varepsilon_0\hbar^2}{me^2}; \qquad E_1^{3D}=-\frac{\hbar^2}{2m(a_B^{3D})^2}.$$

Find the ratios a_B^{2D}/a_B^{3D} and E_1^{2D}/E_1^{3D}. Explain the interaction that makes the ground state energy of the 2-D hydrogen atom so much lower (more negative) than that of the 3-D atom. [3]

(iv) An *exciton* is formed in a semiconductor when a negative charge carrier (electron) becomes bound to a positive charge carrier (hole) by the Coulomb interaction. The resulting system can be described by the hydrogen atom Hamiltonian, only with ε_0 replaced by $\varepsilon_0\varepsilon_r$ where ε_r is the relative permittivity of the semiconductor, and with m replaced by μ, the effective carrier mass. The exciton may be 3-dimensional in a bulk semiconductor, or 2-dimensional when the carriers are confined to one plane by a quantum well. Estimate the Bohr radius and ground state energy in eV for a) a 2-D exciton and b) a 3-D exciton in the semiconductor gallium arsenide (GaAs), for which $\varepsilon_r = 13$ and $\mu = 5 \times 10^{-32}\ \mathrm{kg}$. [4]

(v) Explain why in GaAs at room temperature, $k_B T \sim 25\ \mathrm{meV}$, exciton states can easily be observed in quantum wells, but not in bulk material. [2]

(vi) If the number of 2-D excitons per unit area of a GaAs quantum well becomes so great that their ground state wavefunctions start to overlap, then *screening* is said to reduce the Coulomb interaction between electron and hole, which cease to be bound together in a hydrogenic wavefunction. Estimate the area density of excitons at which this screening effect will appear in a GaAs quantum well. Explain what is meant by screening. [4]

END OF PAPER