SEMESTER 2 EXAMINATIONS 2012-2013

MPhys Synoptic Examination

DURATION 120 MINS

*Answer **all** questions in **Section A**; one **and only one** question in **Section B;** and one **and only one** question in **Section C.***

Section A carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section B carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

Section C carries 1/3 of the total marks for the exam paper and you should aim to spend about 40 mins on it.

An outline marking scheme is shown in brackets to the right of each question.

A Sheet of Physical Constants is provided with this examination paper.

Only University approved calculators may be used.

Section A

A1. The Maxwell curl equation for magnetic field may be written

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

in SI units, where $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{j}$ is the free charge current density and $\boldsymbol{E}$ is the electric field intensity. Demonstrate the dimensional consistency of this equation by showing that each term has dimensions of [V s m^{-3}]. What physical quantity has dimensions of [V s]? [3]

A2. Consider a satellite of mass m, in circular orbit, radius a, around a planet of mass M. Find an expression for the total energy of the satellite as a function of a. A booster rocket fires briefly, imparting an impulse in the direction antiparallel to the instantaneous velocity of the satellite, and reducing its kinetic energy. Describe the subsequent motion of the satellite qualitatively. [4]

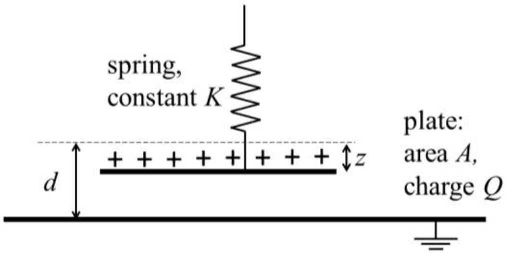
A3. An MRI (magnetic resonance imaging) body scanner uses a superconducting solenoid to generate an intense dc magnetic field within a cylindrical chamber large enough to accommodate a person. Estimate the magnitude in joules of the energy inside the chamber associated with the field when the body scanner is operating. [3]

A4. A reversible heat engine operates between a finite hot reservoir, initially at temperature T_1, and an infinite cold reservoir at constant temperature T_2. The heat capacity of the hot reservoir, C_0, is independent of temperature. Find an expression for the maximum amount of work that may be done by this engine. [5]

A5. Write an expression for the (un-normalised) wavefunction of a free particle of energy E, moving in 1 dimension along the positive x-axis in a region of zero potential. Now suppose that the particle encounters a negative potential step of height $-V_0$. Find an expression for the probability that the particle will be reflected by the step. [5]

SECTION B

B1



A capacitor, shown above, consists of a light flat conducting plate of area A, electrically isolated and suspended under vacuum by a spring above a conducting ground plane held at zero potential. The spring keeps the plate parallel to the ground plane and allows the separation between the plate and the ground plane to vary under the action of a linear restoring force of spring constant K.

(i) A charge $+Q$ is applied to the suspended plate. Explain why the charged plate is attracted towards the ground plane. [2]

(ii) State Gauss's theorem. Show, by reference to a suitable Gauss surface, that a uniform E-field of strength $E = Q/\varepsilon_0 A$ exists between the suspended plate and the ground plane, assuming that distortion of the field at the edges of the plate can be neglected. [2]

(iii) Write down an expression for the total potential energy U of the capacitor, with its suspended plate bearing charge Q and displaced by distance z from its equilibrium position. (The energy density of an electrostatic field in vacuum is $\varepsilon_0 E^2/2$.) [2]

(iv) Explain how the principle of virtual work leads to an expression for the total force F_z acting on the suspended plate. Use the result of (iii) to show that

$$F_z = -\frac{dU}{dz} = -Kz + \frac{Q^2}{2\varepsilon_0 A}.$$ [2]

(v) Show that in equilibrium the displacement of the capacitor plate is related to its charge by $z = Q^2/(2\varepsilon_0 AK)$. [1]

(vi) Prove that the potential difference between the suspended plate and the ground plane is given by

$$V = \frac{Q}{\varepsilon_0 A/d}\left(1 - \frac{Q^2}{2Kd\varepsilon_0 A}\right),$$

where d is the equilibrium distance between the uncharged plate and the ground plane. Sketch a graph of V against Q, explaining the range of Q over which this relationship is valid. [5]

(vii) Prove that the potential difference V between the suspended plate and the ground plane takes a maximum value when $Q = \sqrt{2\varepsilon_0 AKd/3}$ and $z = d/3$. [2]

(viii) Suppose that an external dc voltage source is connected across the capacitor. Explain why stable equilibrium states of the capacitor only exist for

$$-\sqrt{2\varepsilon_0 AKd/3} < Q < \sqrt{2\varepsilon_0 AKd/3}.$$

[4]

TURN OVER

B2. The tip of a nanostructured silicon cantilever is free to oscillate in 1 dimension, with displacement x, under the action of a harmonic restoring force with spring constant K_0. The Hamiltonian operator for this system may be written

$$H = \frac{p^2}{2M} + \frac{1}{2}K_0 x^2$$

where M is the effective mass. Note the value of the definite integral:

$$\int_{-\infty}^{+\infty} \exp(-Bx^2)dx = \sqrt{\frac{\pi}{B}}.$$

(i) Write down the time-independent Schrödinger equation for the silicon cantilever system. [1]

(ii) Show that the time-independent Schrödinger equation in (i) has a solution of the form

$$u_0(x) = A_0 \exp\left(-\frac{1}{2}\alpha_0^2 x^2\right)$$

where A_0 is a real constant and $\alpha_0^4 = MK_0/\hbar^2$. Find the energy eigenvalue of the cantilever in this state. [3]

(iii) Explain why the wavefunction of the cantilever must be normalised to represent its state correctly. State the normalisation condition, and determine the value of A_0 that normalises $u_0(x)$ defined in (ii). [3]

(iv) Explain why it is necessary to cool the cantilever to a temperature $T < T_0$ in order to observe non-classical effects. Estimate a value for T_0, given that

$$\frac{1}{2\pi}\sqrt{\frac{K_0}{M}} = 3\ \mathrm{GHz}.$$

[2]

(v) There occurs a sudden change in the internal structure of the cantilever, after which its spring constant takes the weaker value

$$K = \left(\frac{5}{12}\right)^4 K_0,$$

with no change in effective mass or equilibrium position. Immediately after the change, the wavefunction of the cantilever is $u_0(x)$, given in (ii). Explain why the cantilever is **not** now in a *stationary state*. [4]

(vi) Show that after the change described in (v) a measurement of the energy of the cantilever **may** yield the value

$$\frac{1}{2}\hbar\sqrt{\frac{K}{M}}$$

with probability $\frac{120}{169}$. [5]

(vii) Suggest other possible outcomes of the energy measurement described in (vi). [2]

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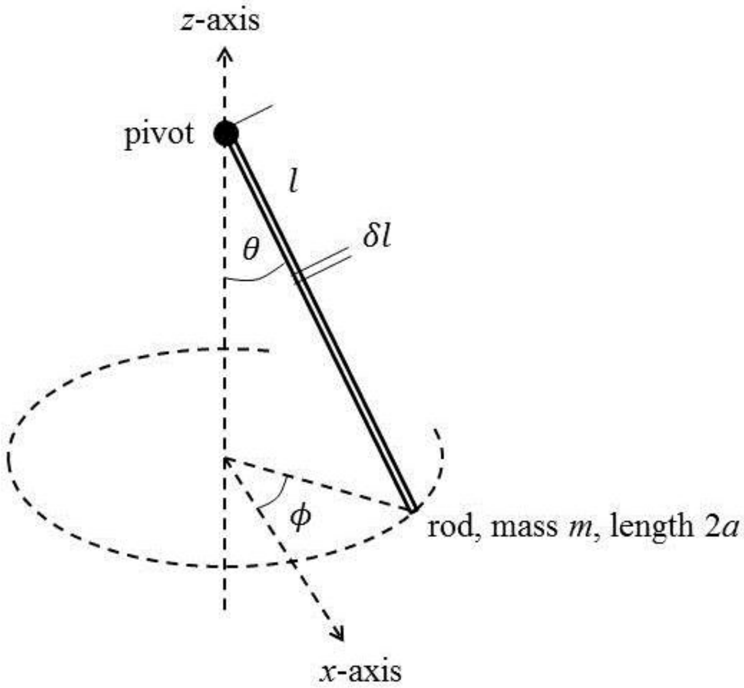
SECTION C

C1



The thin uniform rod of length $2a$ and mass m shown above is freely pivoted at one end, able to rotate about horizontal and vertical axes. In this problem you should neglect air resistance and friction in the pivot. The Euler Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \left(\frac{\partial L}{\partial q_i}\right) = 0$$

describe the motion of a system with Lagrangian $L(q_i, \dot{q}_i)$, where generalised coordinates q_i describe the state of the system, and the dot denotes a time derivative.

(i) A thin slice of the rod of length δl, distance l from the pivot, has x-coordinate $x = l\sin\theta\cos\phi$. Write down expressions for the y- and z-coordinates of this slice. [2]

(ii) Derive an expression for the kinetic energy of the slice δl in terms of l, θ, ϕ, $\dot{\theta}$, and $\dot{\phi}$. [4]

(iii) Hence show that the total kinetic energy of the rod is given by

$$T = \frac{2}{3}ma^2\dot{\theta}^2 + \frac{2}{3}ma^2\dot{\phi}^2\sin^2\theta.$$

State the physical significance of each term in this expression. [2]

(iv) Write down a Lagrangian for the thin rod. Explain what it means to say that ϕ is an **ignorable coordinate** in this problem, and prove that the motion of the rod conserves the quantity $\dot{\phi}\sin^2\theta$. [4]

(v) The rod is set spinning with initial values $\theta = \pi/2$ and $\dot{\phi} = \omega_0$. Show that the subsequent variation of the angle θ is determined by the equation of motion

$$\ddot{\theta} = \omega_0^2\frac{\cos\theta}{\sin^3\theta} - \frac{3g}{4a}\sin\theta.$$

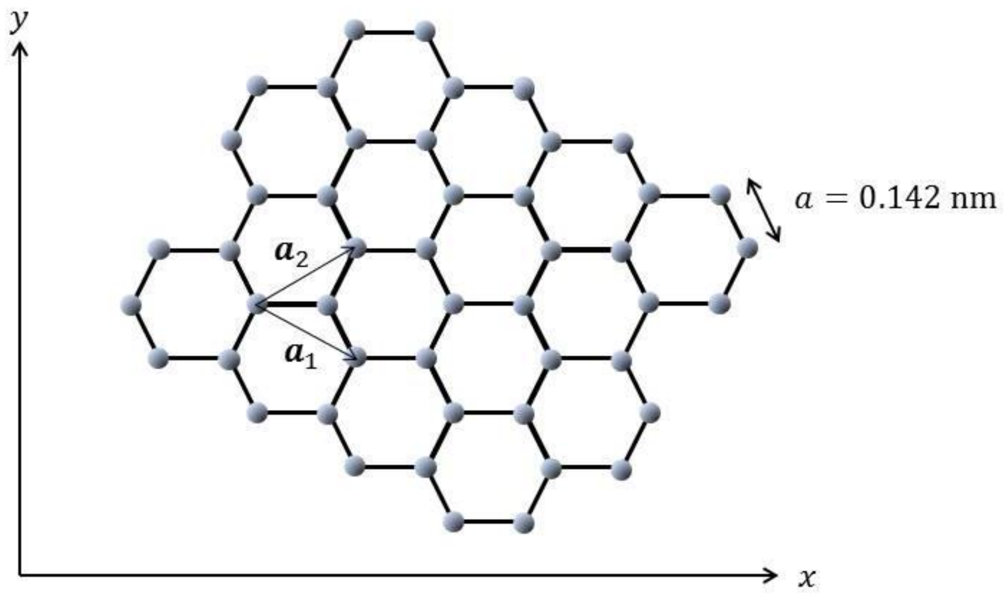
[3]

(vi) Suppose that

$$\omega_0^2 = \frac{3g\sin^4\theta_0}{4a\cos\theta_0}$$

where θ_0 is a constant equal to $\pi/4$. Sketch a graph of $\ddot{\theta}$ against θ. Describe the motion of the rod qualitatively. [5]

C2.



A graphene crystal consists of a single planar layer of carbon atoms arranged in the structure illustrated above, with carbon-carbon bond length $a = 0.142\ \mathrm{nm}$. The primitive lattice translation vectors shown in the diagram are given by

$$\boldsymbol{a}_1 = a\frac{\sqrt{3}}{2}\left(\hat{\boldsymbol{x}}\sqrt{3} - \hat{\boldsymbol{y}}\right) \text{ and } \boldsymbol{a}_2 = a\frac{\sqrt{3}}{2}\left(\hat{\boldsymbol{x}}\sqrt{3} + \hat{\boldsymbol{y}}\right).$$

(i) Define the term **crystal translation vector** used to describe the translational symmetry of a crystal. Write down the position vector of each carbon atom in one unit cell of the structure relative to the nearest space lattice point. [3]

(ii) A monochromatic electromagnetic plane wave, with wavevector $\boldsymbol{k}$ parallel to the z-axis, is incident on a graphene crystal oriented in the x-y plane. The amplitude of a wave diffracted from the crystal with wavevector $\boldsymbol{k}^/$ is proportional to

$$\sum_C f_0 \exp\left(-i\left(\boldsymbol{k}^/ - \boldsymbol{k}\right).\boldsymbol{\rho}_C\right)$$

where f_0 is a constant, $\boldsymbol{\rho}_C$ is the position vector of one carbon atom, and the sum is taken over all carbon atoms in the crystal. Show that the above expression may be factorised into the product of a term that depends only on the space lattice and a term that depends only on the arrangement of atoms in one unit cell. [3]

(iii) Show from the result of part (ii) that a diffracted wave may have non-zero amplitude if

$$\boldsymbol{k}^/_{\text{plane}} = h\boldsymbol{A}_1 + l\boldsymbol{A}_2$$

where $\boldsymbol{k}^/_{\text{plane}}$ is the component of $\boldsymbol{k}^/$ perpendicular to the z-axis, h and l are integers, and $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ are the vectors

$$\boldsymbol{A}_1 = \frac{2\pi}{3a}\left(\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}}\sqrt{3}\right)$$

$$\boldsymbol{A}_2 = \frac{2\pi}{3a}\left(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\sqrt{3}\right).$$

[4]

(iv) A graphene sample consists of a large number of crystal fragments attached to the surface of a glass plate. The z-axis of each fragment is normal to the surface of the plate, with x- and y-axes randomly oriented in the plane of the surface. Explain the shape of the diffracted beams that may be formed when a beam of monochromatic X-rays is transmitted through the sample at normal incidence. [2]

(v) Show that the diffracted beam with $(h, l) = (0, 1)$ is indistinguishable from the beam with $(h, l) = (1, 1)$. [1]

(vi) Show that, when the diffraction experiment described in (iv) is set up with 13.4-keV X-rays, there are diffracted beams formed at angles of 26º and 49º to the z-axis. [4]

(vii) Show that the intensity of the 49º diffracted beam is greater than the intensity of the 26º diffracted beam by a factor of 4. [3]

END OF PAPER